Integrating algebraic fractions 1

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Sometimes the integral of an algebraic fraction can be found by first expressing the algebraic fraction as the sum of its partial fractions. In this unit we will illustrate this idea. We will see that it is also necessary to draw upon a wide variety of other techniques such as completing the square, integration by substitution, using standard forms, and so on.

In order to master the techniques explained here it is vital that you undertake plenty of practice exercises so that they become second nature.

After reading this text, and/or viewing the video tutorial on this topic, you should be able to:

- integrate algebraic fractions by first expressing them in partial fractions
- integrate algebraic fractions by using a variety of other techniques

Contents

1. Introduction

In this section we are going to look at how we can integrate some algebraic fractions. We will be using partial fractions to rewrite the integrand as the sum of simpler fractions which can then be integrated separately. We will also need to call upon a wide variety of other techniques including completing the square, integration by substitution, integration using standard results and so on.

Examples of the sorts of algebraic fractions we will be integrating are

$$\frac{x}{(2-x)(3+x)}, \qquad \frac{1}{x^2+x+1}, \qquad \frac{1}{(x-1)^2(x+1)} \qquad \text{and} \qquad \frac{x^3}{x^2-4}$$

Whilst superficially they may look similar, there are important differences. For example, the denominator of the first contains two linear factors. The second has an irreducible quadratic factor (i.e. it will not factorise), and we consider how to deal with this case in the second video on Integrating by Partial Fractions. The third example contains a factor which is repeated. The fourth is an example of an improper fraction because the degree of the numerator is greater than the degree of the denominator. All of these factors are important in selecting the appropriate way to proceed.

It is also important to consider the *degree* of the numerator and of the denominator. For instance, if we consider the third example, then the degree of its denominator is 3, because when we multiply out $(x-1)^2(x+1)$ the highest power of x is x^3. Also, the degree of the numerator is zero, because we can think of 1 as $1x^0$. So the degree of the numerator is less than the degree of the denominator, and that is the case for the first three of the examples. We call fractions like these *proper fractions*. On the other hand, in the final example, the degree of the numerator is 3 whereas the degree of the denominator is 2. This is called an *improper fraction*.

Key Point

The *degree* of a polynomial expression in x is the highest power of x appearing in the expression. An algebraic fraction where the degree of the numerator is less than the degree of the denominator is called a *proper fraction*. If the degree of the numerator is greater than, or equal to, the degree of the denominator then the fraction is an *improper fraction*.

2. Some preliminary results

To understand the examples which follow you will need to use various techniques which you should have met before. We summarise them briefly here, but you should refer to other relevant material if you need to revise the details.

Partial fractions

A linear factor, $ax+b$ in the denominator gives rise to a partial fraction of the form $\dfrac{A}{ax+b}$.

Repeated linear factors, $(ax+b)^2$ give rise to partial fractions of the form $\dfrac{A}{ax+b}+\dfrac{B}{(ax+b)^2}$.

A quadratic factor ax^2+bx+c gives rise to a partial fraction of the form $\dfrac{Ax+B}{ax^2+bx+c}$.

Integration - standard results

$$\int \frac{f'(x)}{f(x)}\mathrm{d}x = \ln|f(x)|+c \qquad \text{e.g.} \qquad \int \frac{1}{x+1}\mathrm{d}x = \ln|x+1|+c\,,$$

$$\int \frac{1}{a^2+x^2}\mathrm{d}x = \frac{1}{a}\tan^{-1}\frac{x}{a}+c\,.$$

Integration - substitution

To find $\displaystyle\int \frac{1}{(x-1)^2}\mathrm{d}x$, substitute $u=x-1$, $\mathrm{d}u=\left(\dfrac{\mathrm{d}u}{\mathrm{d}x}\right)\mathrm{d}x$ to give

$$\begin{aligned}\int \frac{1}{(x-1)^2}\mathrm{d}x &= \int \frac{1}{u^2}\mathrm{d}u \\ &= \int u^{-2}\mathrm{d}u \\ &= -u^{-1}+c \\ &= -\frac{1}{x-1}+c\,.\end{aligned}$$

3. Algebraic fractions with two linear factors

In this section we will consider how to integrate an algebraic fraction which has the form of a proper fraction with two linear factors in the denominator.

Example

Suppose we want to find

$$\int \frac{x}{(2-x)(x+3)}\,\mathrm{d}x\,.$$

Note that the integrand is a proper fraction (because the degree of the numerator is less than the degree of the denominator), and also that the denominator has two, distinct, linear factors. Therefore the appropriate form for its partial fractions is

$$\frac{x}{(2-x)(x+3)} = \frac{A}{(2-x)}+\frac{B}{(x+3)}$$

where A and B are constants which we shall determine shortly. We add the two terms on the right-hand side together again using a common denominator:

$$\begin{aligned}\frac{x}{(2-x)(x+3)} &= \frac{A}{(2-x)}+\frac{B}{(x+3)} \\ &= \frac{A(x+3)+B(2-x)}{(2-x)(x+3)}\,.\end{aligned}$$

Because the fraction on the left is equal to that on the right for all values of x, and because their denominators are equal, then their numerators too must be equal. So, from just the numerators,

$$x = A(x+3) + B(2-x)\,. \tag{1}$$

We now proceed to find the values of the constants A and B. We can do this in one of two ways, or by mixing the two ways. The first way is to substitute particular values for x. The second way is to separately equate coefficients of constant terms, linear terms, quadratic terms etc. Both of these ways will be illustrated now.

Substitution of particular values for x

Because expression (1) is true for all values of x we can substitute any value we choose for x. In particular, if we let $x = 2$ the second term on the right becomes zero, and everything looks simpler:

$$2 = A(2+3) + 0$$

from which $5A = 2$ and so

$$A = \frac{2}{5}\,.$$

Similarly, substituting $x = -3$ in expression (1) makes the first term zero:

$$-3 = 5B$$

from which

$$B = -\frac{3}{5}\,.$$

Thus the partial fractions are

$$\frac{x}{(2-x)(x+3)} = \frac{2}{5(2-x)} - \frac{3}{5(x+3)}\,.$$

Both of the terms on the right can be integrated:

$$\begin{aligned}\int\left(\frac{2}{5(2-x)} - \frac{3}{5(x+3)}\right)\mathrm{d}x &= -\frac{2}{5}\int\frac{-1}{2-x}\mathrm{d}x - \frac{3}{5}\int\frac{1}{x+3}\mathrm{d}x \\ &= -\frac{2}{5}\ln|2-x| - \frac{3}{5}\ln|x+3| + c\,.\end{aligned}$$

Note that in the first of the two integrals, we have set the numerator to be -1 and compensated for this by writing a minus sign outside the integral. We have done this because the derivative of $2-x$ is -1, so that the integral is in a standard form. So by using partial fractions we have broken down the original integral into two separate integrals which we can then evaluate.

Equating coefficients

A second technique for finding A and B is to equate the coefficients of equivalent terms on each side. First of all we expand the brackets in Equation (1) and collect together like terms:

$$\begin{aligned}x &= Ax + 3A + 2B - Bx \\ &= (A-B)x + 3A + 2B\,.\end{aligned}$$

Equating the coefficients of x on each side:

$$1 = A - B\,. \tag{2}$$

Equating constant terms on each side of this expression gives

$$0 = 3A + 2B\,. \qquad (3)$$

These are two simultaneous equations we can solve to find A and B. Multiplying Equation (2) by 2 gives

$$2 = 2A - 2B\,. \qquad (4)$$

Now, adding (3) and (4) eliminates the B's to give

$$2 = 5A$$

from which $A = \frac{2}{5}$. Also, from (2), $B = A - 1 = \frac{2}{5} - 1 = -\frac{3}{5}$ just as we obtained using the method of substituting specific values for x. Often you will find that a combination of both techniques is efficient.

Example

Suppose we want to evaluate $\displaystyle\int_1^2 \frac{3}{x(x+1)}\mathrm{d}x$.

Note again that the integrand is a proper fraction and also that the denominator has two, distinct, linear factors. Therefore the appropriate form for its partial fractions is

$$\frac{3}{x(x+1)} = \frac{A}{x} + \frac{B}{(x+1)}$$

where A and B are constants we need to find. We add the two terms on the right-hand side together again using a common denominator.

$$\begin{aligned}\frac{3}{x(x+1)} &= \frac{A}{x} + \frac{B}{(x+1)}\\ &= \frac{A(x+1) + Bx}{x(x+1)}\,.\end{aligned}$$

Because the fraction on the left is equal to that on the right for all values of x, and because their denominators are equal, then their numerators too must be equal. So, from just the numerators,

$$3 = A(x+1) + Bx\,.$$

If we substitute $x = 0$ we can immediately find A:

$$3 = A(0+1) + B(0)$$

so that $A = 3$.
If we substitute $x = -1$ we find B:

$$3 = A(-1+1) + B(-1)$$

so that $B = -3$. Then

$$\begin{aligned}\int_1^2 \frac{3}{x(x+1)}\,\mathrm{d}x &= \int_1^2 \left(\frac{3}{x} - \frac{3}{x+1}\right)\mathrm{d}x\\ &= [3\ln|x| - 3\ln|x+1|]_1^2\\ &= (3\ln 2 - 3\ln 3) - (3\ln 1 - 3\ln 2)\\ &= 6\ln 2 - 3\ln 3\\ &= \ln\frac{2^6}{3^3}\\ &= \ln\frac{64}{27}\end{aligned}$$

Exercises 1

1. Find each of the following integrals by expressing the integrand in partial fractions.

(a) $\displaystyle\int \frac{1}{(x+2)(x+1)}\mathrm{d}x$ (b) $\displaystyle\int \frac{x}{(2x+3)(x-4)}\mathrm{d}x$ (c) $\displaystyle\int \frac{3x+2}{(x-1)(x+7)}\mathrm{d}x$

4. Algebraic fractions with a repeated linear factor

When the denominator contains a repeated linear factor care must be taken to use the correct form of partial fractions as illustrated in the following example.

Example

Find $\displaystyle\int \frac{1}{(x-1)^2(x+1)}\mathrm{d}x$.

In this Example there is a repeated factor in the denominator. This is because the factor $x-1$ appears twice, as in $(x-1)^2$. We write

$$\begin{aligned}\frac{1}{(x-1)^2(x+1)} &= \frac{A}{x-1}+\frac{B}{(x-1)^2}+\frac{C}{x+1} \\ &= \frac{A(x-1)(x+1)+B(x+1)+C(x-1)^2}{(x-1)^2(x+1)}\,.\end{aligned}$$

As before, the fractions on the left and the right are equal for all values of x. Their denominators are equal and so we can equate the numerators:

$$1 = A(x-1)(x+1)+B(x+1)+C(x-1)^2\,. \qquad (1)$$

Substituting $x=1$ in Equation (1) gives $1=2B$, from which $B=\frac{1}{2}$.

Substituting $x=-1$ gives $1=4C$ from which $C=\frac{1}{4}$.

Knowing B and C, substitution of any other value for x will give the value of A. For example, if we let $x=0$ we find

$$1=-A+B+C$$

and so

$$1=-A+\frac{1}{2}+\frac{1}{4}$$

from which $A=-\frac{1}{4}$. Alternatively, we could have expanded the right-hand side of Equation (1), collected like terms together and equated coefficients. This would have yielded the same values for A, B and C.

The integral becomes

$$\begin{aligned}\int \frac{1}{(x-1)^2(x+1)}\mathrm{d}x &= \int\left(-\frac{1}{4(x-1)}+\frac{1}{2(x-1)^2}+\frac{1}{4(x+1)}\right)\mathrm{d}x \\ &= -\frac{1}{4}\ln|x-1|-\frac{1}{2(x-1)}+\frac{1}{4}\ln|x+1|+c\,.\end{aligned}$$

Using the laws of logarithms this can be written in the following alternative form if required:

$$\frac{1}{4}\ln\left|\frac{x+1}{x-1}\right|-\frac{1}{2(x-1)}+c\,.$$

Exercises 2

1. Integrate each of the following by expressing the integrand in partial fractions.

(a) $\displaystyle\int \frac{1}{(x+3)^2(x-1)}\mathrm{d}x$ (b) $\displaystyle\int \frac{2x+1}{(x+2)^2(x+1)}\mathrm{d}x$ (c) $\displaystyle\int \frac{x+1}{x(x-7)^2}\mathrm{d}x$.

5. Dealing with improper fractions

When the degree of the numerator is greater than or equal to the degree of the denominator the fraction is said to be improper. In such cases it is first necessary to carry out long division as illustrated in the next Example.

Example

Find $\displaystyle\int \frac{x^3}{x^2-4}\mathrm{d}x$.

The degree of the numerator is greater than the degree of the numerator. This fraction is therefore improper. We can divide the denominator into the numerator using long division of fractions:

$$\begin{array}{r|l} & \;\; x \\ \cline{2-2} x^2-4 \,) & x^3 \\ & x^3 - 4x \\ \cline{2-2} & \; 4x \end{array}$$

so that

$$\frac{x^3}{x^2-4} = x + \frac{4x}{x^2-4}\,.$$

Note that the denominator of the second term on the right hand side is the **difference of two squares** and can be factorised as $x^2-4 = (x-2)(x+2)$. So,

$$\begin{aligned} \frac{4x}{x^2-4} = \frac{4x}{(x-2)(x+2)} &= \frac{A}{x-2} + \frac{B}{x+2} \\ &= \frac{A(x+2)+B(x-2)}{(x-2)(x+2)}\,. \end{aligned}$$

As before, the fractions on the left and on the right are equal for all values of x. Their denominators are the same, and so too must be their numerators. So we equate the numerators to give

$$4x = A(x+2) + B(x-2)\,.$$

Choosing $x = 2$ we find $8 = 4A$ so that $A = 2$. Choosing $x = -2$ gives $-8 = -4B$ so that $B = 2$. So with these values of A and B the integral becomes

$$\begin{aligned} \int \frac{x^3}{x^2-4}\,\mathrm{d}x &= \int \left(x + \frac{2}{x-2} + \frac{2}{x+2} \right) \mathrm{d}x \\ &= \frac{x^2}{2} + 2\ln|x-2| + 2\ln|x+2| + c\,. \end{aligned}$$

Exercises 3

1. Use long division and partial fractions to find the following integrals.

(a) $\displaystyle\int \frac{x^3+1}{1-x^2}\mathrm{d}x$ (b) $\displaystyle\int \frac{x^2+3x+3}{x+1}\mathrm{d}x$

(c) $\displaystyle\int \frac{7x-6}{x-1}\mathrm{d}x$ (d) $\displaystyle\int \frac{7x^2+16x-19}{x^2+2x-3}\mathrm{d}x$

Answers

Exercises 1

1. (a) $\ln|x+1| - \ln|x+2| + c$ (b) $\frac{3}{22}\ln|2x+3| + \frac{4}{11}\ln|x-4| + c$ (c) $\frac{5}{8}\ln|x-1| + \frac{19}{8}\ln|x+7| + c$.

Exercises 2

1. (a) the partial fractions are: $-\dfrac{1}{4}\dfrac{1}{(x+3)^2} - \dfrac{1}{16}\dfrac{1}{x+3} + \dfrac{1}{16}\dfrac{1}{x-1}$;

the integral is $\dfrac{1}{4}\dfrac{1}{x+3} - \dfrac{1}{16}\ln|x+|) + \dfrac{1}{16}\ln|x-1| + C$

(b) the partial fractions are $\dfrac{3}{(x+2)^2} + \dfrac{1}{x+2} - \dfrac{1}{x+1}$;

the integral is $-\dfrac{3}{x+2} + \ln|x+2| - \ln|x+1| + C$.

(c) the partial fractions are $\dfrac{1}{49x} + \dfrac{8}{7(x-7)^2} - \dfrac{1}{49(x-7)}$;

the integral is $\dfrac{1}{49}\ln|x| - \dfrac{8}{7(x-7)} - \dfrac{1}{49}\ln|x-7| + C$.

Exercises 3

1. (a) the partial fractions are $-x - \dfrac{1}{x-1}$;

the integral is $-\dfrac{x^2}{2} - \ln|x-1| + C$.

(b) the partial fractions are $x+2+\dfrac{1}{x+1}$;

the integral is $\dfrac{x^2}{2} + 2x + \ln|x+1| + C$.

(c) the partial fractions are $7 + \dfrac{1}{x-1}$;

the integral is $7x + \ln|x-1| + C$.

(d) the partial fractions are $7 + \dfrac{1}{x+3} + \dfrac{1}{x-1}$;

the integral is $7x + \ln|x+3| + \ln|x-1| + C$.

