Differentiation by taking logarithms

In this unit we look at how we can use logarithms to simplify certain functions before we differentiate them.

In order to master the techniques explained here it is vital that you undertake plenty of practice exercises so that they become second nature.

After reading this text, and/or viewing the video tutorial on this topic, you should be able to:

- use logarithms to simplify functions before differentiation

Contents

1. Introduction

In this unit we look at how we can use logarithms to simplify certain functions before we differentiate them. To start off, we remind you about logarithms themselves.

If $y = \ln x$, the natural logarithm function, or the log to the base e of x, then

$$\frac{dy}{dx} = \frac{1}{x}$$

You should be familiar already with this result. If not, you should learn it, and you can also refer to the unit on differentiation of the logarithm and exponential functions.

We will also need to differentiate $y = \ln f(x)$, where $f(x)$ is some function of x. In this case,

$$\frac{dy}{dx} = \frac{f'(x)}{f(x)}$$

You should learn this result too3

Key Point

$$\text{if } y = \ln x \text{ then } \frac{dy}{dx} = \frac{1}{x} \qquad \text{if } y = \ln f(x) \text{ then } \frac{dy}{dx} = \frac{f'(x)}{f(x)}$$

We will also make use of the following laws of logarithms:

Key Point

Laws of logarithms:

$$\log A + \log B = \log AB, \quad \log A - \log B = \log \frac{A}{B}, \quad m \log A = \log A^m$$

These laws can be used to simplify logarithms of any base, but the same base must be used throughout a particular calculation.

2. Some examples

Example

Suppose we want to differentiate $y = \ln(3x^4 + 7)^5$.

Using the laws of logarithms we can write y as

$$y = 5\ln(3x^4 + 7)$$

This is more straightforward to differentiate because we have $y = \ln f(x)$ with $f(x) = 3x^4 + 7$. The rule given in the Key Point on page 2 tells us that

$$\frac{dy}{dx} = 5 \times \frac{12x^3}{3x^4 + 7} = \frac{60x^3}{3x^4 + 7}$$

What started out as quite a complicated problem was simplified using the laws of logarithms before we carried out the differentiation.

Example

Suppose we want to differentiate $y = \ln\left(\dfrac{1 - 3x}{1 + 2x}\right)$.

In this example we have the log of a quotient. We can use the law $\ln\frac{A}{B} = \ln A - \ln B$ to rewrite this as

$$y = \ln(1 - 3x) - \ln(1 + 2x)$$

The two functions on the right are easy to differentiate using the Key Point on page 2:

$$\frac{dy}{dx} = \frac{-3}{1 - 3x} - \frac{2}{1 + 2x}$$

We can write the answer as a single term by writing them both over a common denominator:

$$\frac{dy}{dx} = \frac{-3(1 + 2x) - 2(1 - 3x)}{(1 - 3x)(1 + 2x)} = \frac{-3 - 6x - 2 + 6x}{(1 - 3x)(1 + 2x)} = -\frac{5}{(1 - 3x)(1 + 2x)}$$

Example

Suppose we want to differentiate $y = x^{\sin x}$.

The problem here is that the function $\sin x$ appears as a power. By taking logarithms of both sides, and using the laws of logarithms we can avoid this as follows:

Taking logs of both sides gives:
$$\ln y = \ln x^{\sin x}$$

and using the third of the laws given on page 2:

$$\ln y = \sin x \, \ln x$$

The term on the right is a product of $\sin x$ and $\ln x$ and we will be able to use the product rule to differentiate it.

The term on the left, being $\ln y$, needs to be differentiated implicitly. Recall that

$$\frac{d}{dx}(\ln y) = \frac{d}{dy}(\ln y) \times \frac{dy}{dx} = \frac{1}{y}\frac{dy}{dx}$$

So,

$$\frac{1}{y}\frac{dy}{dx} = \sin x \frac{1}{x} + \ln x \, \cos x = \frac{\sin x + x \ln x \cos x}{x}$$

so that

$$\begin{aligned}
\frac{dy}{dx} &= y\left(\frac{\sin x + x \ln x \cos x}{x}\right) \\
&= x^{\sin x}\left(\frac{\sin x + x \ln x \cos x}{x}\right)
\end{aligned}$$

since we recall that $y = x^{\sin x}$.

Example

Suppose we wish to differentiate $y = \dfrac{(1-2x)^3}{\sqrt{1+x^2}}$.

We proceed by taking logarithms of both sides to give:

$$\ln y = \ln\frac{(1-2x)^3}{\sqrt{1+x^2}}$$

We then use the laws of logarithms to rewrite the right hand side as

$$\ln y = \ln(1-2x)^3 - \ln\sqrt{(1+x^2)}$$

With a further application of the laws of logarithms, and noting that $\sqrt{1+x^2} = (1+x^2)^{1/2}$, we obtain

$$\ln y = 3\ln(1-2x) - \frac{1}{2}\ln(1+x^2)$$

Differentiating, and remembering to differentiate the left hand side implicitly

$$\begin{aligned}
\frac{1}{y}\frac{dy}{dx} &= 3 \times \frac{-2}{1-2x} - \frac{1}{2} \times \frac{2x}{1+x^2} \\
&= \frac{-6}{1-2x} - \frac{x}{1+x^2} \\
&= \frac{-6(1+x^2) - x(1-2x)}{(1-2x)(1+x^2)} \\
&= \frac{-6 - 6x^2 - x + 2x^2}{(1-2x)(1+x^2)} \\
&= \frac{-6 - x - 4x^2}{(1-2x)(1+x^2)}
\end{aligned}$$

so that

$$\frac{dy}{dx} = y \times \frac{-6 - x - 4x^2}{(1 - 2x)(1 + x^2)}$$

$$= \frac{(1 - 2x)^3}{\sqrt{1 + x^2}} \times \frac{-6 - x - 4x^2}{(1 - 2x)(1 + x^2)}$$

This can be simpified to

$$\frac{dy}{dx} = \frac{-(6 + x + 4x^2)(1 - 2x)^2}{(1 + x^2)^{3/2}}$$

From these examples we have seen how using logarithms to write certain functions in alternative forms can help when we need to differentiate them.

Exercises

1. Find the derivative of each of the following functions

   a) $\ln(x^3 + 1)$ b) $\ln(\sin x)$ c) $\ln((x^3 + 2x + 1)^4)$ d) $\ln\left(\dfrac{x^2 + 1}{x - 1}\right)$ e) $\ln(x^2 \sin x)$

2. By taking logs and using implicit differentiation, find the derivatives of the following functions

   a) $\dfrac{1 - x^2}{\sqrt{1 + 2x}}$ b) x^x c) $x^{\ln x}$ d) $\dfrac{(1 + x^2)^3}{(1 - x^3)^2}$ e) x^{ax+b}

Answers

1. a) $\dfrac{3x^2}{x^3 + 1}$ b) $\cot x$ c) $\dfrac{4(3x^2 + 2)}{x^3 + 2x + 1}$ d) $\dfrac{x^2 - 2x - 1}{(x^2 + 1)(x - 1)}$ e) $\dfrac{2}{x} + \cot x$

2. a) $-\dfrac{1 + 2x + 3x^2}{(1 + 2x)^{3/2}}$ b) $x^x(\ln x + 1)$ c) $2x^{\ln x - 1} \ln x$

   d) $\dfrac{6x(1 + x)(1 + x^2)^2}{(1 - x^3)^3}$ e) $x^{ax+b-1}(ax(\ln x + 1) + b)$