Exponential and logarithm functions

Exponential functions and logarithm functions are important in both theory and practice. In this unit we look at the graphs of exponential and logarithm functions, and see how they are related.

In order to master the techniques explained here it is vital that you undertake plenty of practice exercises so that they become second nature.

After reading this text, and/or viewing the video tutorial on this topic, you should be able to:

- specify for which values of a the exponential function $f(x) = a^x$ may be defined,

- recognize the domain and range of an exponential function,

- identify a particular point which is on the graph of every exponential function,

- specify for which values of a the logarithm function $f(x) = \log_a x$ may be defined,

- recognize the domain and range of a logarithm function,

- identify a particular point which is on the graph of every logarithm function,

- understand the relationship between the exponential function $f(x) = \mathrm{e}^x$ and the natural logarithm function $f(x) = \ln x$.

Contents

1. Exponential functions

Consider a function of the form $f(x) = a^x$, where $a > 0$. Such a function is called an *exponential* function. We can take three different cases, where $a = 1$, $0 < a < 1$ and $a > 1$.

If $a = 1$ then

$$f(x) = 1^x = 1.$$

So this just gives us the constant function $f(x) = 1$.

What happens if $a > 1$? To examine this case, take a numerical example. Suppose that $a = 2$.

$$f(x) = 2^x$$

$$f(0) = 2^0 = 1$$

$$f(1) = 2^1 = 2 \qquad\qquad f(-1) = 2^{-1} = 1/2^1 = \tfrac{1}{2}$$

$$f(2) = 2^2 = 4 \qquad\qquad f(-2) = 2^{-2} = 1/2^2 = \tfrac{1}{4}$$

$$f(3) = 2^3 = 8 \qquad\qquad f(-3) = 2^{-3} = 1/2^3 = \tfrac{1}{8}$$

We can put these results into a table, and plot a graph of the function.

x	$f(x)$
-3	$\frac{1}{8}$
-2	$\frac{1}{4}$
-1	$\frac{1}{2}$
0	1
1	2
2	4
3	8
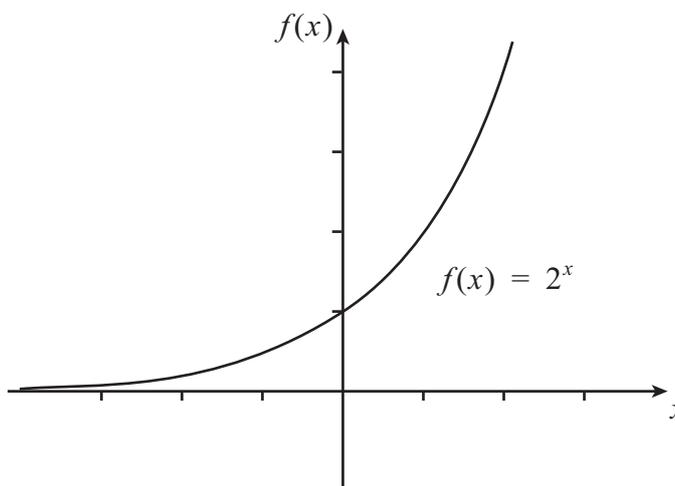
This example demonstrates the general shape for graphs of functions of the form $f(x) = a^x$ when $a > 1$.

What is the effect of varying a? We can see this by looking at sketches of a few graphs of similar functions.
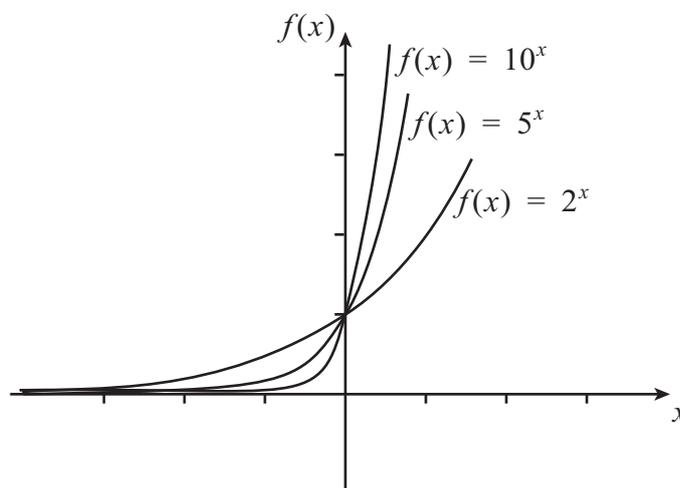
The important properties of the graphs of these types of functions are:

- $f(0) = 1$ for all values of a. This is because $a^0 = 1$ for any value of a.

- $f(x) > 0$ for all values of a. This is because $a > 0$ implies $a^x > 0$.

What happens if $0 < a < 1$? To examine this case, take another numerical example. Suppose that $a = \frac{1}{2}$.

$$f(x) = \left(\tfrac{1}{2}\right)^x$$

$$f(0) = \left(\tfrac{1}{2}\right)^0 = 1$$

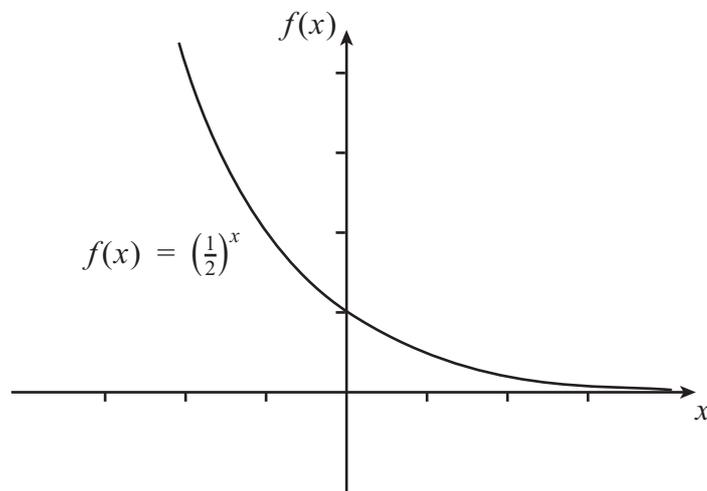$$f(1) = \left(\tfrac{1}{2}\right)^1 = \left(\tfrac{1}{2}\right) \qquad\qquad f(-1) = \left(\tfrac{1}{2}\right)^{-1} = \left(\tfrac{2}{1}\right)^1 = 2$$

$$f(2) = \left(\tfrac{1}{2}\right)^2 = \left(\tfrac{1}{4}\right) \qquad\qquad f(-2) = \left(\tfrac{1}{2}\right)^{-2} = \left(\tfrac{2}{1}\right)^2 = 4$$

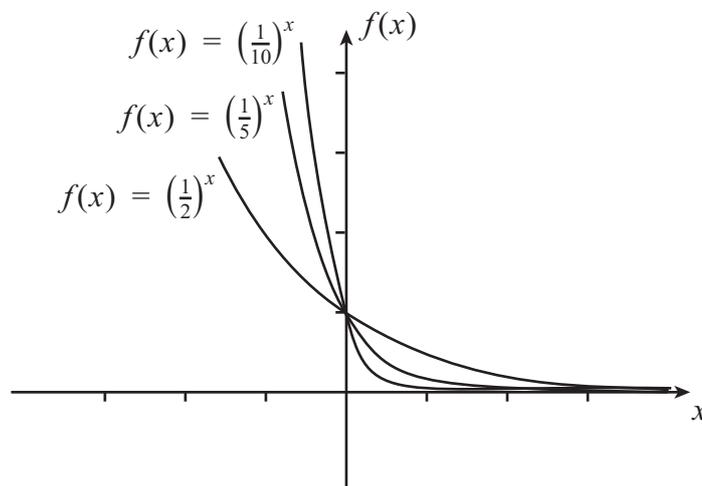$$f(3) = \left(\tfrac{1}{2}\right)^3 = \left(\tfrac{1}{8}\right) \qquad\qquad f(-3) = \left(\tfrac{1}{2}\right)^{-3} = \left(\tfrac{2}{1}\right)^3 = 8$$

We can put these results into a table, and plot a graph of the function.

x	$f(x)$
-3	8
-2	4
-1	2
0	1
1	$\frac{1}{2}$
2	$\frac{1}{4}$
3	$\frac{1}{8}$

This example demonstrates the general shape for graphs of functions of the form $f(x) = a^x$ when $0 < a < 1$.

What is the effect of varying a? Again we can see by looking at sketches of a few graphs of similar functions.

The important properties of the graphs of these types of functions are:

- $f(0) = 1$ for all values of a. This is because $a^0 = 1$ for any value of a.

- $f(x) > 0$ for all values of a. This is because $a > 0$ implies $a^x > 0$.

Notice that these properties are the same as when $a > 1$. One interesting thing that you might have spotted is that $f(x) = (\frac{1}{2})^x = 2^{-x}$ is a reflection of $f(x) = 2^x$ in the $f(x)$ axis, and that $f(x) = (\frac{1}{5})^x = 5^{-x}$ is a reflection of $f(x) = 5^x$ in the $f(x)$ axis.
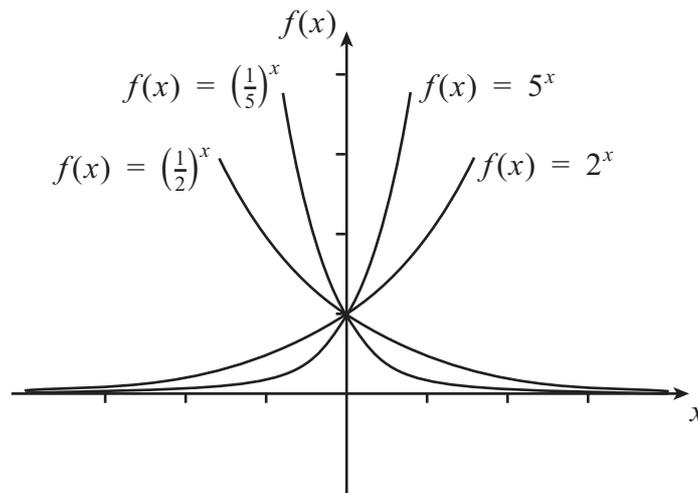
In general, $f(x) = (1/a)^x = a^{-x}$ is a reflection of $f(x) = a^x$ in the $f(x)$ axis.

A particularly important example of an exponential function arises when $a = e$. You might recall that the number e is approximately equal to 2.718. The function $f(x) = e^x$ is often called 'the' exponential function. Since e > 1 and $1/e < 1$, we can sketch the graphs of the exponential functions $f(x) = e^x$ and $f(x) = e^{-x} = (1/e)^x$.
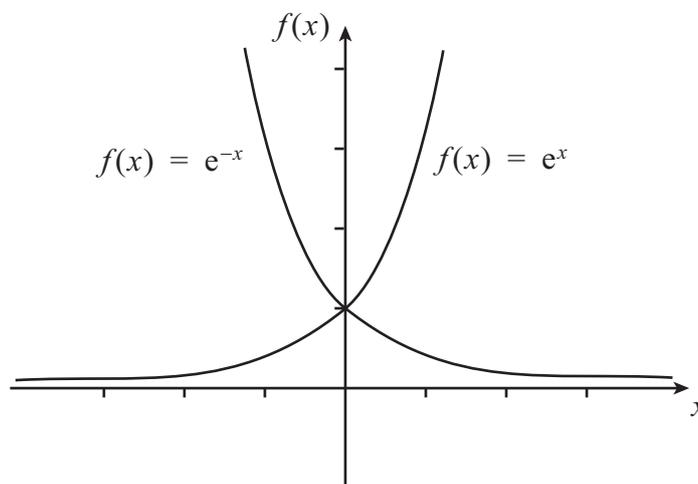
Key Point

A function of the form $f(x) = a^x$ (where $a > 0$) is called an exponential function.

The function $f(x) = 1^x$ is just the constant function $f(x) = 1$.

The function $f(x) = a^x$ for $a > 1$ has a graph which is close to the x-axis for negative x and increases rapidly for positive x.

The function $f(x) = a^x$ for $0 < a < 1$ has a graph which is close to the x-axis for positive x and increases rapidly for decreasing negative x.

For any value of a, the graph always passes through the point $(0, 1)$. The graph of $f(x) = (1/a)^x = a^{-x}$ is a reflection, in the vertical axis, of the graph of $f(x) = a^x$.

A particularly important exponental function is $f(x) = e^x$, where $e = 2.718\ldots$. This is often called 'the' exponential function.

2. Logarithm functions

We shall now look at logarithm functions. These are functions of the form $f(x) = \log_a x$ where $a > 0$. We do not consider the case $a = 1$, as this will not give us a valid function.

What happens if $a > 1$? To examine this case, take a numerical example. Suppose that $a = 2$. Then

$$f(x) = \log_2 x \qquad \text{means} \qquad 2^{f(x)} = x.$$

An important point to note here is that, regardless of the argument, $2^{f(x)} > 0$. So we shall consider only positive arguments.

$$
\begin{aligned}
f(1) &= \log_2 1 & \text{means} \quad & 2^{f(1)} = 1 & \text{so} \quad & f(1) = 0 \\
f(2) &= \log_2 2 & \text{means} \quad & 2^{f(2)} = 2 & \text{so} \quad & f(2) = 1 \\
f(4) &= \log_2 4 & \text{means} \quad & 2^{f(4)} = 4 & \text{so} \quad & f(4) = 2 \\
f(\tfrac{1}{2}) &= \log_2(\tfrac{1}{2}) & \text{means} \quad & 2^{f(\frac{1}{2})} = \tfrac{1}{2} = 2^{-1} & \text{so} \quad & f(\tfrac{1}{2}) = -1 \\
f(\tfrac{1}{4}) &= \log_2(\tfrac{1}{4}) & \text{means} \quad & 2^{f(\frac{1}{4})} = \tfrac{1}{4} = 2^{-2} & \text{so} \quad & f(\tfrac{1}{4}) = -2
\end{aligned}
$$

We can put these results into a table, and plot a graph of the function.
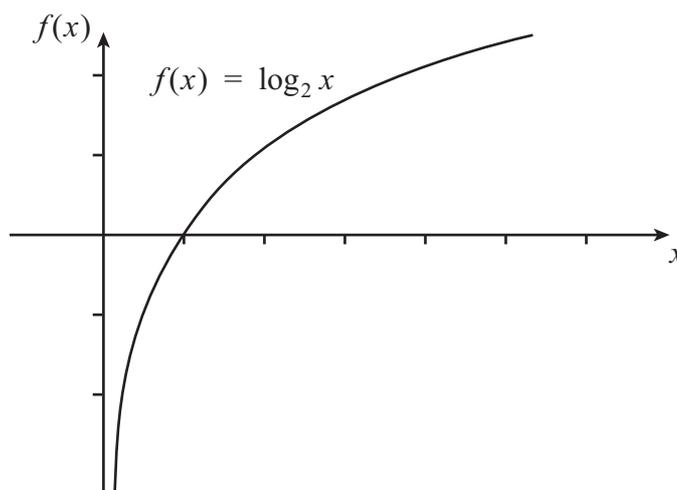
x	$f(x)$
$\frac{1}{4}$	-2
$\frac{1}{2}$	-1
1	0
2	1
4	2

This example demonstrates the general shape for graphs of functions of the form $f(x) = \log_a x$ when $a > 1$.

What is the effect of varying a? We can see by looking at sketches of a few graphs of similar functions. For the special case where $a = \mathrm{e}$, we often write $\ln x$ instead of $\log_\mathrm{e} x$.
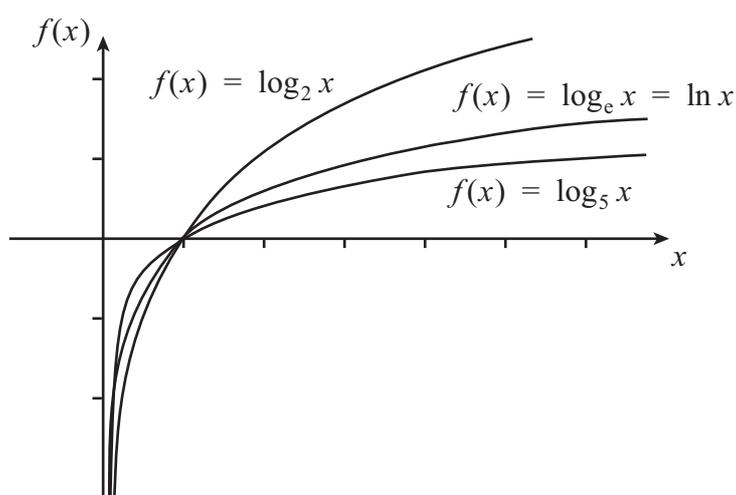
The important properties of the graphs of these types of functions are:

- $f(1) = 0$ for all values of a;

- we must have $x > 0$ for all values of a.

What happens if $0 < a < 1$? To examine this case, take another numerical example. Suppose that $a = \frac{1}{2}$. Then

$$f(x) = \log_{1/2} x \qquad \text{means} \qquad \left(\tfrac{1}{2}\right)^{f(x)} = x\,.$$

An important point to note here is that, regardless of the argument, $\left(\tfrac{1}{2}\right)^{f(x)} > 0$. So we shall consider only positive arguments.

$$f(x) = \left(\tfrac{1}{2}\right)^x$$

$$f(1) = \log_{1/2} 1 \quad \text{means} \quad \left(\tfrac{1}{2}\right)^{f(1)} = 1 \qquad\qquad \text{so} \quad f(1) = 0$$
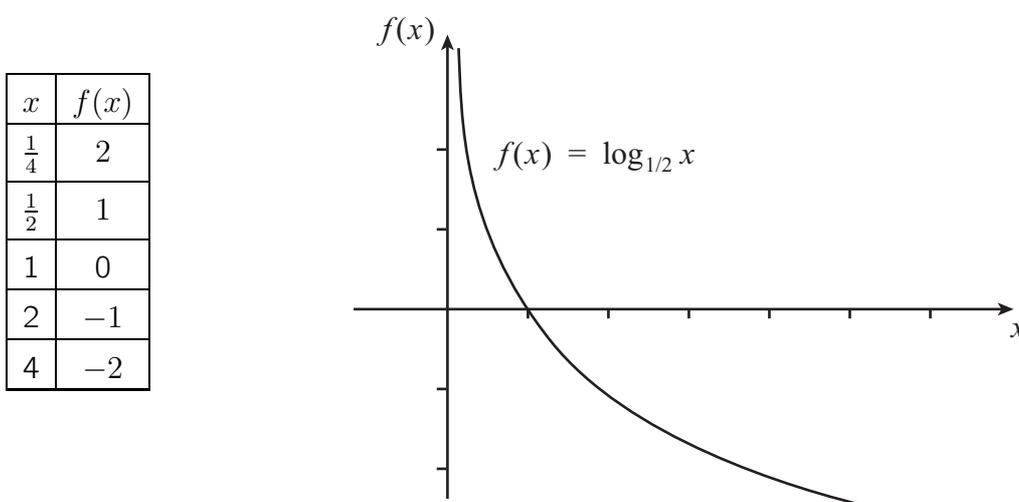
$$f(2) = \log_{1/2} 2 \quad \text{means} \quad \left(\tfrac{1}{2}\right)^{f(2)} = 2 = \left(\tfrac{1}{2}\right)^{-1} \quad \text{so} \quad f(2) = -1$$

$$f(4) = \log_{1/2} 4 \quad \text{means} \quad \left(\tfrac{1}{2}\right)^{f(4)} = 4 = \left(\tfrac{1}{2}\right)^{-2} \quad \text{so} \quad f(4) = -2$$

$$f(\tfrac{1}{2}) = \log_{1/2}(\tfrac{1}{2}) \quad \text{means} \quad \left(\tfrac{1}{2}\right)^{f(\frac{1}{2})} = \tfrac{1}{2} \qquad\qquad \text{so} \quad f(\tfrac{1}{2}) = 1$$
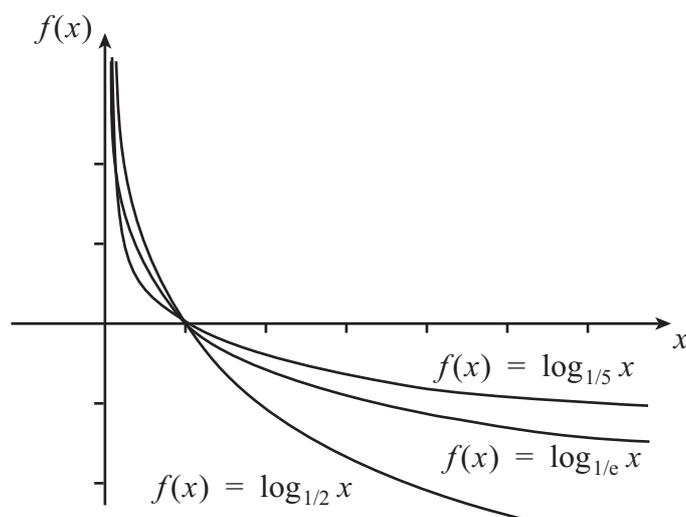
$$f(\tfrac{1}{4}) = \log_{1/2}(\tfrac{1}{4}) \quad \text{means} \quad \left(\tfrac{1}{2}\right)^{f(\frac{1}{4})} = \tfrac{1}{4} = \left(\tfrac{1}{2}\right)^{2} \quad \text{so} \quad f(\tfrac{1}{4}) = 2$$

We can put these results into a table, and plot a graph of the function.

x	$f(x)$
$\frac{1}{4}$	2
$\frac{1}{2}$	1
1	0
2	−1
4	−2

This example demonstrates the general shape for graphs of functions of the form $f(x) = \log_a x$ when $0 < a < 1$.

What is the effect of varying a? Again we can see by looking at sketches of a few graphs of similar functions.

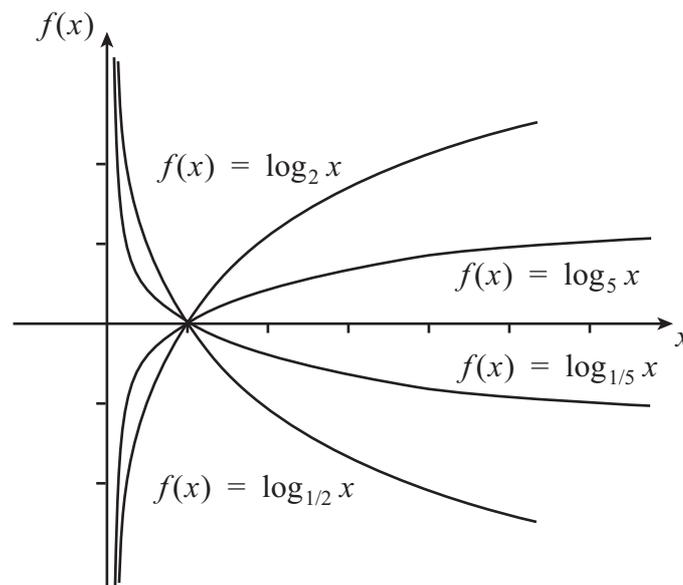The important properties of the graphs of these types of functions are:

- $f(1) = 0$ for all values of a;

- we must have $x > 0$ for all values of a.

An interesting thing that you might well have spotted is that

$$f(x) = \log_{1/5} x \text{ is a reflection of } f(x) = \log_5 x \text{ in the } x\text{-axis}$$

and

$$f(x) = \log_{1/2} x \text{ is a reflection of } f(x) = \log_2 x \text{ in the } x\text{-axis.}$$

Generally,

$$f(x) = \log_{1/a} x \text{ is a reflection of } f(x) = \log_a x \text{ in the } x\text{-axis.}$$

Key Point

A function of the form $f(x) = \log_a x$ (where $a > 0$ and $a \neq 1$) is called a logarithm function.

The function $f(x) = \log_a x$ for $a > 1$ has a graph which is close to the negative $f(x)$-axis for $x < 1$ and increases slowly for positive x.

The function $f(x) = \log_a x$ for $0 < a < 1$ has a graph which is close to the positive $f(x)$-axis for $x < 1$ and decreases slowly for positive x.

For any value of a, the graph always passes through the point $(1, 0)$. The graph of $f(x) = \log_{1/a} x$ is a reflection, in the horizontal axis, of the graph of $f(x) = \log_a x$.

A particularly important logarithm function is $f(x) = \log_e x$, where $e = 2.718\ldots$. This is often called the natural logarithm function, and written $f(x) = \ln x$.

3. The relationship between exponential functions and logarithm functions

We can see the relationship between the exponential function $f(x) = e^x$ and the logarithm function $f(x) = \ln x$ by looking at their graphs.
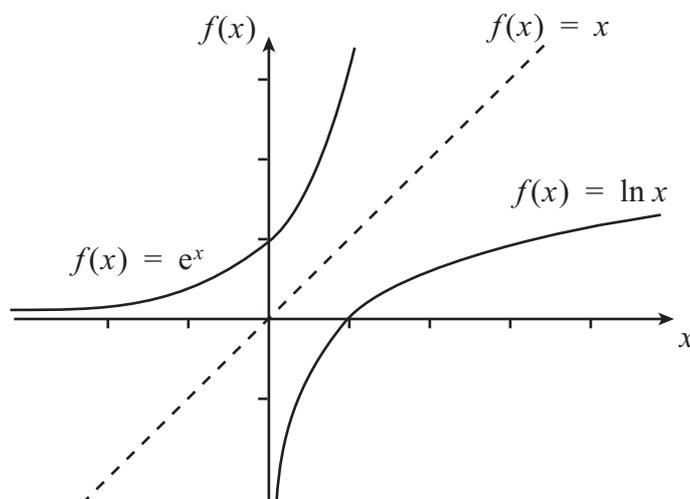
You can see straight away that the logarithm function is a reflection of the exponential function in the line represented by $f(x) = x$. In other words, the axes have been swapped: x becomes $f(x)$, and $f(x)$ becomes x.

Key Point

The exponential function $f(x) = e^x$ is the inverse of the logarithm function $f(x) = \ln x$.

## Exercises

1. Sketch the graph of the function $f(x) = a^x$ for the following values of a, on the same axes.

   (a) $a = 3$ (b) $a = 6$ (c) $a = 1$ (d) $a = \frac{1}{3}$ (e) $a = \frac{1}{6}$

2. Sketch the graph of the function $f(x) = \log_a x$ for the following values of a, on the same axes.

   (a) $a = 3$ (b) $a = 6$ (c) $a = \frac{1}{3}$ (d) $a = \frac{1}{6}$

3. For each of the following pairs of functions, state whether the graphs are related by a reflection in the x-axis, a reflection in the $f(x)$-axis, a reflection in the line $f(x) = x$, a reflection in the line $f(x) = -x$, or that the graphs are not related by any of these reflections.
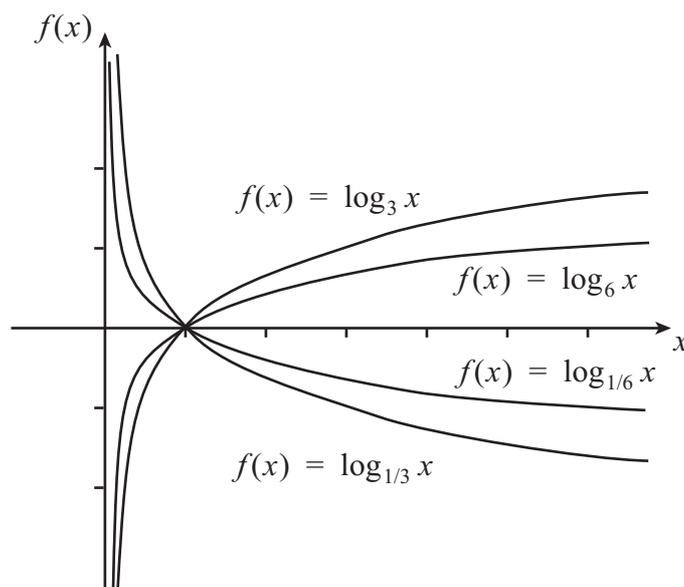
(a) $f(x) = 3^x$ and $f(x) = \left(\frac{1}{3}\right)^x$

(b) $f(x) = \log_6 x$ and $f(x) = 6^x$

(c) $f(x) = \log_6 x$ and $f(x) = \left(\frac{1}{6}\right)^x$

(d) $f(x) = \log_{1/3} x$ and $f(x) = \log_3 x$

(e) $f(x) = \left(\frac{1}{3}\right)^x$ and $f(x) = \left(\frac{1}{6}\right)^x$

Answers

1.
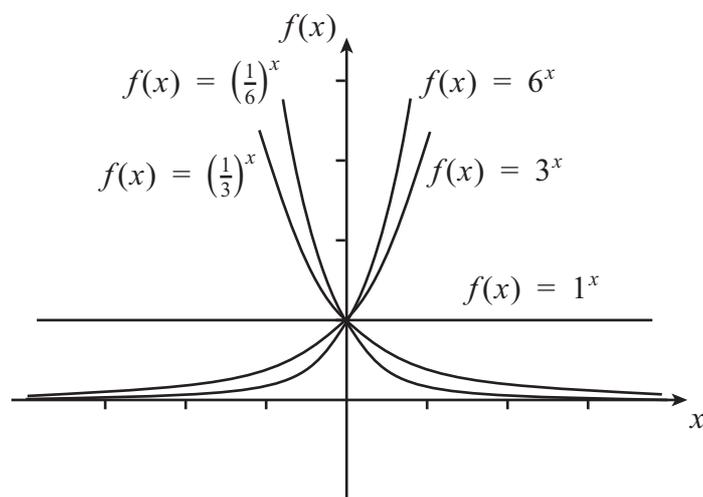
2.

3.
- (a) Reflect in the $f(x)$-axis
- (b) Reflect in the line $f(x) = x$
- (c) Not related by any of these reflections
- (d) Reflect in the x-axis
- (e) Not related by any of these reflections