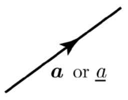
1. Vectors

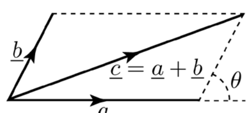
Force, velocity and acceleration which involve both a magnitude and direction, are **vectors**. A vector is written using a bold typeface, $\boldsymbol{a}$, or an underline $\underline{a}$. It is represented pictorially by a **directed line segment** as shown.



The length of the line segment represents the vector's magnitude. Its orientation, together with the arrow shown, gives the direction of the vector. The magnitude of a vector $\underline{a}$ is written $|\underline{a}|$ or simply a. A **unit vector** has magnitude 1. $-\underline{a}$ has the magnitude of $\underline{a}$ but is opposite in direction.

Addition: The parallelogram rule defines addition of two vectors. $\underline{c} = \underline{a} + \underline{b} = \underline{b} + \underline{a}$.

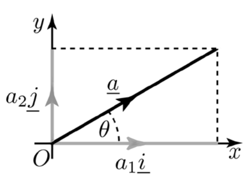
$$c^2 = a^2 + b^2 + 2\,a\,b\,\cos\theta$$



where θ is the angle between $\underline{a}$ and $\underline{b}$, as shown. $\underline{c}$ is called the **resultant** of $\underline{a}$ and $\underline{b}$.

Rectangular Components: Let $\underline{i}$ be a unit vector in the direction of the positive x axis and $\underline{j}$ be a unit vector in the direction of the positive y axis. In two dimensions the vector $\underline{a}$ can be written as the sum of two rectangular vector components: $\underline{a} = a_1\underline{i} + a_2\underline{j}$ or $\underline{a} = (a_1, a_2)$.

The scalar components a_1 and a_2 are given by $a_1 = a\cos\theta$, $a_2 = a\sin\theta$, where θ is the angle $\underline{a}$ makes with the positive x axis. Any vector can be replaced by its rectangular vector components, starting at the same point.



Using Pythagoras' theorem it follows that $|\underline{a}| = \sqrt{a_1^2 + a_2^2}$.

In a natural extension to three dimensions we can write

$$\underline{a} = a_1\underline{i} + a_2\underline{j} + a_3\underline{k} \quad \text{and} \quad |\underline{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$$

$\underline{k}$ is a unit vector in the direction of the positive z axis.

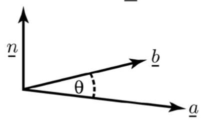
Scalar (dot) product & Vector (cross) product:

If $\underline{a} = a_1\underline{i} + a_2\underline{j} + a_3\underline{k}$ and $\underline{b} = b_1\underline{i} + b_2\underline{j} + b_3\underline{k}$ then

$$\underline{a}\cdot\underline{b} = |\underline{a}|\,|\underline{b}|\cos\theta$$

$$\underline{a}\cdot\underline{b} = a_1b_1 + a_2b_2 + a_3b_3$$

$$\underline{a}\times\underline{b} = |\underline{a}|\,|\underline{b}|\sin\theta\,\underline{n}$$



Here θ is the angle between $\underline{a}$ and $\underline{b}$, and $\underline{n}$ is a unit vector perpendicular to the plane containing $\underline{a}$ and $\underline{b}$ in a sense defined by the right-hand screw rule.

$$\underline{a}\times\underline{b} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$= (a_2b_3 - a_3b_2)\underline{i} - (a_1b_3 - a_3b_1)\underline{j} + (a_1b_2 - a_2b_1)\underline{k}$$

2. Newton's Laws of Motion and Gravitation

Newton's first law of motion: A body will remain at rest or continue its uniform motion in a straight line unless compelled to change by forces acting on it. It follows from this that when a body is in equilibrium the resultant force, $\underline{R} = (R_x, R_y, R_z)$, of all the forces acting on it, is zero. Thus

$$\underline{R} = \underline{0},\ R_x = 0,\ R_y = 0,\ R_z = 0$$

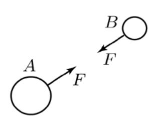
where R_x, R_y and R_z are the net sums of the x, y and z scalar components of the forces, respectively.

Newton's second law of motion: If a body of mass m is moving with velocity $\underline{v}$, and so has **momentum** $m\underline{v}$, then the rate of change of momentum of the body is directly proportional to the resultant applied force, $\underline{F}$, acting on it: $\underline{F} = \dfrac{d}{dt}(m\underline{v})$. For a body with acceleration $\underline{a}$ and of *constant* mass m, this becomes $\underline{F} = m\dfrac{d\underline{v}}{dt} = m\underline{a}$.

This vector equation is equivalent to the scalar equations: $F_x = ma_x$, $F_y = ma_y$, $F_z = ma_z$ where $\underline{F} = (F_x, F_y, F_z)$ and $\underline{a} = (a_x, a_y, a_z)$.

Newton's third law of motion:

To every action there is an equal and opposite reaction. Thus forces come in pairs when bodies interact. Whenever body A exerts a force, $\underline{F}$, of magnitude F, on body B, B exerts a force, $-\underline{F}$, on body A.



Newton's Law of Universal Gravitation: Every body in the universe attracts every other body with a force which is directly proportional to the product of the masses and inversely proportional to the square of the distance between them. Thus $F_g = G\dfrac{m_1m_2}{r^2}$ where F_g is the magnitude of the gravitational force on either body, m_1 and m_2 are their masses, r is the distance between them. G is called the gravitational constant. Its accepted value is $G = 6.673\times10^{-11}\,\mathrm{m^3\,kg^{-1}s^{-2}}$.

3. Units

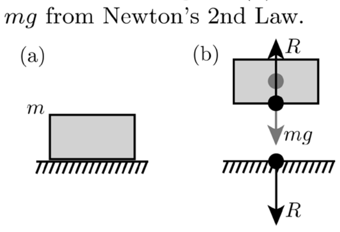
The SI system uses the following units:

Quantity	Unit	Symbol
Mass	kilogram	kg
Length	metre	m
Time	second	s
Force	newton	N $(1\,\mathrm{N} = 1\,\mathrm{kg\,m\,s^{-2}})$
Work	joule	J $(1\,\mathrm{J} = 1\,\mathrm{N\,m})$
Power	watt	W $(1\,\mathrm{W} = 1\,\mathrm{J\,s^{-1}})$
Velocity	metre per second	$\mathrm{m\,s^{-1}}$
Acceleration	metre per second per second	$\mathrm{m\,s^{-2}}$
Energy	joule	J
Momentum	newton second	N s
Impulse	newton second	N s
Angular Velocity/ Angular Frequency	radians per second	rad $\mathrm{s^{-1}}$

4. Forces (1)

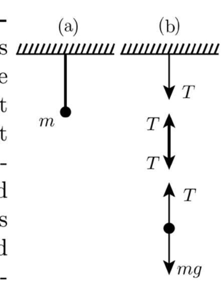
Weight: The weight of a body, of mass m, is defined to be the force, $\underline{W}$, with which it is attracted to the Earth. Its magnitude, W, is given by the Law of Gravitation with $r \approx R$ (radius of the Earth) as $W = \dfrac{mGM}{R^2}$, where M is the mass of the Earth. Weight is also given by Newton's Second Law. For a body falling under gravity with constant acceleration g 'close to the Earth's surface' $\underline{W} = m\underline{g}$ and so $g = \dfrac{GM}{R^2}$; $g \approx 9.81\,\mathrm{m\,s^{-2}}$.

Reaction: A block, of mass m, rests on a horizontal surface, as shown in diagram (a). The block and the surface interact, exerting on each other equal and opposite normal reactions of magnitude R. A *separated body diagram* for the block is shown in diagram (b). Since the block is at rest $R = mg$ from Newton's 2nd Law.



Tension: (i) Light, inextensible strings.

A mass m hangs in equilibrium on the end of an inextensible string attached to a ceiling, diagram (a). The **tension** at any point of the string equals the force exerted at that point. The string is said to be 'light' if its weight is negligible compared to the weight mg, and the tension, T, is then constant along its length. A separated body diagram (b) shows the forces acting on the mass and the string, and shows the tension exerted on the ceiling. From Newton's 2nd Law $T = mg$.

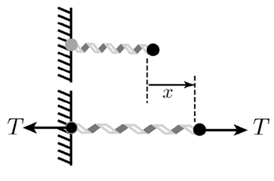


Tension: (ii) Elastic strings or springs (Hooke's Law).

Hooke showed that, provided the extension is not too great, the tension, T, in an elastic string or spring is directly proportional to the extension, x, and inversely proportional to its natural length, L:

$T = \dfrac{\lambda x}{L}$ where λ is Young's modulus of elasticity, or

$T = kx$ where $k = \dfrac{\lambda}{L}$ is called the 'spring stiffness'.



12. Impulse & Momentum

Linear momentum, $\underline{p}$, of a body of mass, m, with velocity, $\underline{v}$, is a vector quantity defined as $\underline{p} = m\underline{v}$.

Impulse: If a constant force, $\underline{F}$, acts over a time, t, on the body then the impulse of the force is defined as Impulse $= \underline{F}\,t$. Impulse is a vector quantity. The unit of impulse is the same as the unit of momentum.

Relationship between momentum and impulse: If a force acts on a body over a time t, the impulse of the force equals the final momentum minus the initial momentum. For the case of a constant force,

$$\underline{F}t = m\underline{v} - m\underline{u}$$

Principle of conservation of linear momentum: When no resultant external force acts on a system of interacting (colliding) particles the total momentum of the system remains constant.

The collision of two bodies: An **elastic** collision is one in which the total kinetic energy is conserved. An **inelastic** collision is one in which the total kinetic energy always decreases. Consider the collision between two spheres moving in the same line.

Let

m_1, m_2 = the masses of the two spheres

u_1, u_2 = the velocities before collision

v_1, v_2 = the velocities after collision

$v_a = u_1 - u_2$ = the speed of approach

$v_s = v_2 - v_1$ = the speed of separation

In a collision v_a and v_s are connected by the relation

$$v_s = e\,v_a, \quad \text{or} \quad v_2 - v_1 = e(u_1 - u_2)$$

where $0 \le e \le 1$ and is called the **coefficient of restitution**.

In an elastic collision, $e = 1$. For an elastic collision

$$m_1v_1 + m_2v_2 = m_1u_1 + m_2u_2$$

$$\frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 = \frac{1}{2}m_1u_1^2 + \frac{1}{2}m_2u_2^2$$

In the case of spheres having the same mass $(m_1 = m_2)$

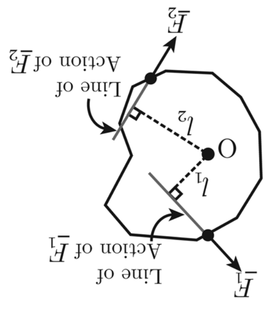
$$u_2 = v_1, \quad u_1 = v_2$$

which means the spheres exchange velocities.

In a 'perfectly inelastic' collision, where the bodies coalesce, $e = 0$. Then $v_1 = v_2$; there is no rebound, as shown.

13. Rigid bodies

Consider an axis perpendicular to the plane of the paper and passing through O. The rigid body is acted upon by the forces $\underline{F}_1$ and $\underline{F}_2$, lying in the plane. $\underline{F}_1$, $\underline{F}_2$ produce anti-clockwise/clockwise rotation about the axis, respectively. By convention, anti-clockwise rotation is taken as positive. The **moments** of $\underline{F}_1$ and $\underline{F}_2$ about the axis through O are defined by



$$\Gamma_1 = +F_1\,l_1 \qquad \Gamma_2 = -F_2\,l_2$$

where l_1 and l_2 are the perpendicular distances of the lines of action of $\underline{F}_1$ and $\underline{F}_2$ from O. The line of action of a force is a line with the same orientation as the force and which passes through its point of action.

For rigid bodies there are two necessary conditions for equilibrium:

First condition: When a body is in equilibrium the resultant force, $\underline{R} = (R_x, R_y, R_z)$, of all the forces acting on it, is zero. (This condition also applies to particles.) Thus

$$\underline{R} = \underline{0} \qquad R_x = 0 \qquad R_y = 0 \qquad R_z = 0$$

where R_x, R_y and R_z are the net sums of the x, y and z scalar components of the forces, respectively.

Second condition: When a body is in equilibrium the sum of the moments, about any arbitrary axis, is zero:

$$\Sigma\,\Gamma = 0$$

Centre of mass: This is the point in a body such that an external force produces an acceleration just as though the whole mass were concentrated there. Let $(\overline{x}, \overline{y}, \overline{z})$ be the coordinates of the centre of mass of a system of particles, each of mass m_1, m_2, …, and centres of mass located at (x_1, y_1, z_1), (x_2, y_2, z_2), …. Then

$$\overline{x} = \frac{\Sigma m_ix_i}{\Sigma m_i} \qquad \overline{y} = \frac{\Sigma m_iy_i}{\Sigma m_i} \qquad \overline{z} = \frac{\Sigma m_iz_i}{\Sigma m_i}$$

from which

$$\Sigma m_i(x_i - \overline{x}) = \Sigma m_i(y_i - \overline{y}) = \Sigma m_i(z_i - \overline{z}) = 0$$

Then the sum of moments about an axis through the centre of mass is zero. Symmetry can be useful in finding the centre of mass. The centre of mass of a homogeneous sphere, circular disk or rectangular plate is at its centre.

Written by Dr. Carol Robinson[1], Dr. Tony Croft[1], & Prof. Mike Savage[2]
with additional comments by Dr. Marie Bassford[3]
Images produced by Paul Newman[1]

[1] Mathematics Education Centre [2] Department of Physics & Astronomy University of Leeds
[3] Engineering Education Centre

43799 D&PS Jan11

mathcentre

For the help you need to support your course

Mechanics Facts & Formulae

mathcentre is a project offering students and staff free resources to support the transition from school mathematics to university mathematics in a range of disciplines.

www.mathcentre.ac.uk

This leaflet has been produced in conjunction with and is distributed by the Higher Education Academy Maths, Stats & OR Network.

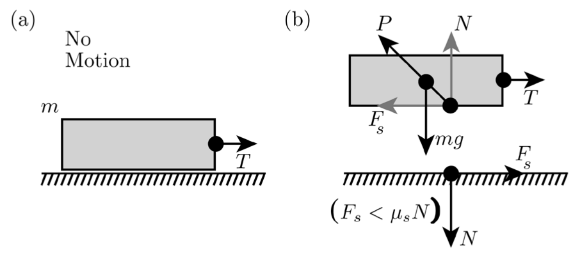
For more copies contact the Network at info@mathstore.ac.uk

sigma centre for excellence in mathematics & statistics support

The Higher Education Academy Maths, Stats & OR Network

5. Forces (2)

Friction: The force which prevents, or tries to prevent, the slipping or sliding of two surfaces in contact is called **friction**. When the surface of one body slides over another, each body exerts a frictional force on the other, parallel to the surfaces. The frictional force on each body is opposite to the direction of its motion. Frictional forces may also act when there is no relative motion, as shown.



A cord is attached to a block of weight $\underline{W} = m\underline{g}$ and the tension, $\underline{T}$, in the cord is such that the block remains at rest (diagram (a)). Diagram (b) is the corresponding separated body diagram. $\underline{P}$ is the force exerted on the block by the surface. $\underline{N}$ and $\underline{F}_s$ are the components of $\underline{P}$, normal to and parallel to the surface. $\underline{F}_s$ is called the force of static friction. From Newton's 2nd law,

$$\underline{N} = -\underline{W} \quad \text{and} \quad \underline{F}_s = -\underline{T}$$

with corresponding scalar forms

$$N = W \quad \text{and} \quad F_s = T$$

As $\underline{T}$ is increased, a limiting value is reached after which the block starts to move. Thus there is a certain maximum value which $\underline{F}_s$ can have. The magnitude of this maximum value depends on the normal force $\underline{N}$ and a useful empirical law is

$$F_s(\max) = \mu_s N$$

where μ_s is called the coefficient of static friction. The magnitude of the actual force of static friction can take any value between 0 and $F_s(\max)$. Thus

$$F_s \leq \mu_s N$$

As soon as sliding begins, the friction force decreases. This new friction force, $\underline{F}_k$, also depends on the normal force. The empirical law used is

$$F_k = \mu_k N$$

where μ_k is the coefficient of sliding (or kinetic) friction. The values of μ_s and μ_k depend on the nature of the two surfaces which are in contact.

6. Kinematics: Rectilinear Motion

A **particle** is a body which can be modelled as a point mass in a given context. For example, for the motion of the planets about the Sun, then the Sun, Earth, etc., can be regarded as particles.

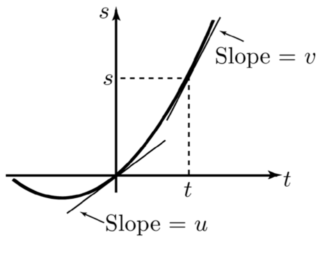
Kinematics is the study of the motions of particles and rigid bodies without any consideration of the forces required to produce these motions. Rectilinear motion is concerned with the motion of a single particle along a straight line.

Constant acceleration: The equations of motion are

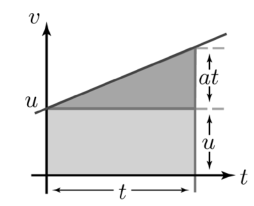
$$v = u + at$$
$$s = \frac{1}{2}(u+v)t \quad \text{or} \quad s = ut + \frac{1}{2}at^2$$
$$v^2 = u^2 + 2as$$

where a is the (constant) acceleration, t represents time, v is the velocity at time t, u is the velocity at $t = 0$, s is the displacement at time t, and $s = 0$ at $t = 0$. These equations are obtained from $\dfrac{dv}{dt} = a$ and $\dfrac{ds}{dt} = v$.

The curve shown here is the **displacement-time graph** for motion with constant acceleration. The slope of the tangent at time t equals the velocity at time t.



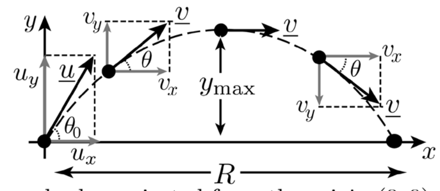
The diagram here shows a **velocity-time graph** for rectilinear motion with constant acceleration. The area under a velocity-time graph equals the displacement. The gradient of the line represents the acceleration.



Non-constant acceleration: Here the acceleration, a, is a function of time, t. As for constant acceleration, the equations of motion are found by integrating $\dfrac{dv}{dt} = a(t)$ and $\dfrac{ds}{dt} = v$.

7. Motion in a Plane: Projectiles

Any object that is given an initial velocity and which subsequently follows a path determined by the gravitational force acting on it and by the frictional resistance of the atmosphere is called a projectile.



Consider a body projected from the origin $(0,0)$ with initial velocity $\underline{u} = (u_x, u_y)$ at an angle of departure θ_0. At any later time t, let (x, y) be its coordinates, and $\underline{v} = (v_x, v_y)$ its velocity. θ is the angle $\underline{v}$ makes with the horizontal, measured in an anti-clockwise sense. If we neglect air resistance, the motion of the projectile can be described as a combination of horizontal motion with constant velocity and vertical motion with constant acceleration. This follows from Newton's Second Law which, in component form, gives

$$\frac{dv_x}{dt} = 0 \text{ and so } v_x = u_x = u\cos\theta_0$$
$$\frac{dv_y}{dt} = -g \text{ and so } v_y = u_y - gt = u\sin\theta_0 - gt$$

The speed v and angle θ are then given by

$$v = \sqrt{v_x^2 + v_y^2} \qquad \tan\theta = \frac{v_y}{v_x}$$

The coordinates of the projectile are

$$x = u_x t = (u\cos\theta_0)t$$
$$y = u_y t - \frac{1}{2}gt^2 = (u\sin\theta_0)t - \frac{1}{2}gt^2$$

The two preceding equations give the equation of the trajectory in terms of the parameter t. By eliminating t, the equation in terms of x and y is

$$y = (\tan\theta_0)x - \frac{g}{2u^2\cos^2\theta_0}x^2$$

This last equation can be recognised as the equation of a parabola. At the highest point, the vertical velocity, v_y, is zero, and hence the time to reach the highest point is $\dfrac{u\sin\theta_0}{g}$. The highest point is given by $y_{\max} = \dfrac{u^2\sin^2\theta_0}{2g}$.

The horizontal range, R, is the horizontal distance from the starting point to the point at which the projectile returns to its original elevation, and at which therefore $y = 0$. Hence $R = \dfrac{u^2\sin 2\theta_0}{g}$. The maximum range occurs when $\sin 2\theta_0 = 1$, i.e. when $\theta_0 = \dfrac{\pi}{4}$ and then $R_{\max} = \dfrac{u^2}{g}$.

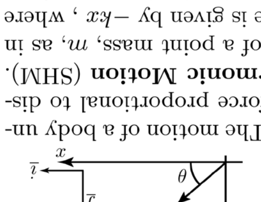
8. Motion of a particle (1)

If a moving particle P has cartesian coordinates (x, y) its position vector is $\underline{r} = x\underline{i} + y\underline{j}$ where both x and y are functions of time, t. Since $\underline{i}$ and $\underline{j}$ are constant vectors, it follows, by differentiating, that its velocity and acceleration vectors are $\underline{v} = \dot{\underline{r}} = \dot{x}\underline{i} + \dot{y}\underline{j}$ and $\underline{a} = \ddot{\underline{r}} = \ddot{x}\underline{i} + \ddot{y}\underline{j}$. Here the dot $\cdot$ denotes a derivative with respect to t. In polar coordinates (r, θ), $x = r\cos\theta$ and $y = r\sin\theta$. Define unit vectors radially and tangentially as $\underline{e}_r$ and $\underline{e}_\theta$. Then

$$\underline{e}_r = \cos\theta\,\underline{i} + \sin\theta\,\underline{j} \qquad \underline{e}_\theta = -\sin\theta\,\underline{i} + \cos\theta\,\underline{j}$$
$$\dot{\underline{e}}_r = -\sin\theta\,\dot{\theta}\,\underline{i} + \cos\theta\,\dot{\theta}\,\underline{j} = \dot{\theta}\,\underline{e}_\theta$$
$$\dot{\underline{e}}_\theta = -\cos\theta\,\dot{\theta}\,\underline{i} - \sin\theta\,\dot{\theta}\,\underline{j} = -\dot{\theta}\,\underline{e}_r$$

It then follows that

$$\underline{r} = r\underline{e}_r$$
$$\dot{\underline{r}} = \dot{r}\underline{e}_r + r\dot{\theta}\underline{e}_\theta$$
$$\ddot{\underline{r}} = (\ddot{r} - r\dot{\theta}^2)\underline{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\underline{e}_\theta$$



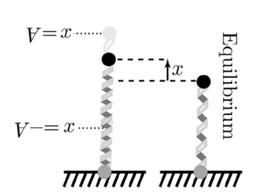
$\dot{\theta}$ is the **angular velocity**, ω.

Simple Harmonic Motion (SHM): The motion of a body under the influence of a restoring force proportional to displacement is called **Simple Harmonic Motion** (SHM). For the one-dimensional motion of a point mass, m, as in a mass-spring oscillator, the force is given by $-kx$, where k is the spring constant and x is the displacement from equilibrium; the equation of SHM is

$$-kx = m\frac{d^2x}{dt^2}$$

or $$\frac{d^2x}{dt^2} + \omega^2 x = 0$$

where $\omega^2 = k/m$.



The solution of this equation is

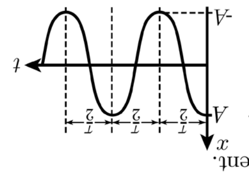
$$x(t) = C\cos\omega t + D\sin\omega t = A\cos(\omega t + \epsilon)$$

for arbitrary constants C and D, and so

$$v(t) = -\omega A\sin(\omega t + \epsilon)$$

It follows that $v^2 = \omega^2(A^2 - x^2)$. Here t is time, the amplitude, A, is the maximum value of $|x|$, v is the velocity, and ϵ is the initial phase angle. The period, τ, is the time for a complete oscillation. The frequency, f, is the number of oscillations per unit time. ω is the angular frequency given by $\omega = \dfrac{2\pi}{\tau} = 2\pi f = \sqrt{\dfrac{k}{m}}$. The graph shows $x(t)$ for the case $\epsilon = 0$. The initial position of the particle is its maximum positive displacement.

The maximum speed occurs when $x = 0$, i.e. at the centre of the oscillation. The acceleration is maximum when the displacement x is maximum.



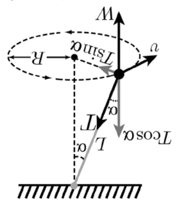
9. Motion of a particle (2)

Circular motion: In circular motion, r is constant and so $\dot{r} = \ddot{r} = 0$. The velocity and acceleration vectors are then

$$\dot{\underline{r}} = r\dot{\theta}\underline{e}_\theta \qquad \ddot{\underline{r}} = -r\dot{\theta}^2\underline{e}_r + r\ddot{\theta}\underline{e}_\theta$$

When the circular motion is uniform the speed, $r\dot{\theta}$, is constant, v say, and so $\ddot{\theta} = 0$. Then $\dot{\underline{r}} = v\underline{e}_\theta$, $\ddot{\underline{r}} = -\dfrac{v^2}{r}\underline{e}_r$. So, if a particle of mass m moves uniformly in a circle of radius r, with speed v, the radial acceleration has magnitude v^2/r and is directed inward along the radius.

The conical pendulum: A particle of mass m revolves in a horizontal circle with constant speed v at the end of a cord of length L. The cord makes an angle α with the vertical. The radius of the circle is $R = L\sin\alpha$. Hence $v = (L\sin\alpha)\omega$, where $\omega = \dot{\theta}$ is the angular speed of motion in the horizontal circle.

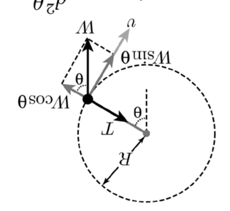


The forces exerted on the body are its weight, of magnitude W, and the tension in the cord which resolves into horizontal and vertical components of magnitudes $T\sin\alpha$ and $T\cos\alpha$ resp. The body has no vertical acceleration and the radial acceleration has magnitude v^2/R. From Newton's 2nd law vertically and radially

$$T\cos\alpha - W = 0 \quad \text{and} \quad T\sin\alpha = \frac{mv^2}{R}$$

Then $\tan\alpha = \dfrac{v^2}{Rg}$ and $\cos\alpha = \dfrac{g/L}{\omega^2}$. Motion arises only if $\cos\alpha < 1$, that is $\omega^2 > g/L$. If $\omega^2 < g/L$ then $\alpha = 0$.

Motion in a vertical circle: Consider a small body of mass m attached to a cord of length R and whirling in a vertical circle about O. The cord makes an angle θ, measured anti-clockwise from the downward vertical. The motion is circular but not uniform. The forces acting on the body are its weight, $\underline{W} = m\underline{g}$, and the tension $\underline{T}$ in the cord. The radial acceleration has magnitude v^2/R where $v = R\dot{\theta}$ (not necessarily constant).

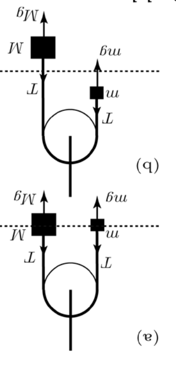


From Newton's 2nd law in the radial direction the magnitude of the tension in the cord is $T = m\left(\dfrac{v^2}{R} + g\cos\theta\right)$. From Newton's 2nd law in the tangential direction $-mg\sin\theta = mR\ddot{\theta}$.

Writing $\dot{\theta}$ as ω, $\dfrac{d^2\theta}{dt^2} = \dfrac{d\omega}{dt} = \omega\dfrac{d\omega}{d\theta}$, this becomes $-mg\sin\theta = mR\omega\dfrac{d\omega}{d\theta}$ which by integrating gives the energy equation $\dfrac{1}{2}mV^2 = \dfrac{1}{2}mv^2 + mgR(1-\cos\theta)$ where V is the speed when $\theta = 0$. The critical speed below which the cord becomes slack ($T = 0$) at its highest point (where $\theta = \pi$) is $v_c = \sqrt{Rg}$.

10. Motion of connected particles

Two masses m and M, with $M \geq m$, are connected by a light, inextensible string which passes over a pulley. When the pulley is smooth the tension, T, in the string is the same throughout its length. Because the string is inextensible the accelerations of the two masses have the same magnitude a. In motion both masses will have the same speed and travel the same distances. The system is released from rest in the position shown in (a).



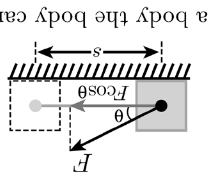
When in motion, (b), from Newton's 2nd law: $T - mg = ma$, and $Mg - T = Ma$, from which

$$a = \left(\frac{M-m}{M+m}\right)g, \qquad T = \frac{2Mmg}{M+m}$$

11. Work, Energy and Power

Work done by a constant force: The figure below represents a body moving in a horizontal direction. A constant force, $\underline{F}$, at an angle θ to the direction of motion, is exerted on the body. The **work done**, W, by the force, when its point of application undergoes a displacement $\underline{s}$, is $W = \underline{F} \cdot \underline{s} = (F\cos\theta)s$. Work is a scalar quantity.

If the component of the force is in the same / opposite direction as the displacement, the work done is positive / negative respectively. If the force is at right angles to the displacement the work done is zero.



Energy: When a force does work on a body the body can gain or lose energy.

Kinetic Energy: K.E. is due to a body's motion. When a body of mass m moves with speed v its K.E. is defined as K.E. $= \frac{1}{2}mv^2$. The change in the K.E. of a rigid body is equal to the work done by the external forces on the body.

Potential Energy: P.E. is due to a body's position.

Gravitational Potential Energy is the product of the weight, mg, of a body and the height, h, of its centre of gravity above a reference level. So P.E. (gravitational) $= mgh$.

Conservation of Total Mechanical Energy: When the only force acting on the body is the gravitational force, the total mechanical energy, which is the sum of the kinetic and potential energies of the body, is conserved.

Power and Velocity: The rate at which work is done is called the **power**. If a constant force $\underline{F}$ is exerted on a body which moves with speed v in the direction of the force, then the power is $P = Fv$.