

Differentiation of the sine and cosine functions from first principles

In order to master the techniques explained here it is vital that you undertake plenty of practice exercises so that they become second nature.

After reading this text, and/or viewing the video tutorial on this topic, you should be able to:

- differentiate the function $\sin x$ from first principles
- differentiate the function $\cos x$ from first principles

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1. Introduction

In this unit we look at how to differentiate the functions $f(x) = \sin x$ and $f(x) = \cos x$ from first principles. We need to remind ourselves of some familiar results.

The derivative of $f(x)$.

The definition of the derivative of a function $y = f(x)$ is

$$\frac{dy}{dx} = \lim_{\delta x \rightarrow 0} \frac{f(x + \delta x) - f(x)}{\delta x}$$

Two trigonometric identities.

We will make use of the trigonometric identities

$$\sin C - \sin D = 2 \cos \frac{C + D}{2} \sin \frac{C - D}{2}$$

$$\cos C - \cos D = -2 \sin \left(\frac{C + D}{2} \right) \sin \left(\frac{C - D}{2} \right)$$

The limit of the function $\frac{\sin \theta}{\theta}$.

As θ (measured in radians) approaches zero, the function $\frac{\sin \theta}{\theta}$ tends to 1. We write this as

$$\lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} = 1$$

This result can be justified by choosing values of θ closer and closer to zero and examining the behaviour of $\frac{\sin \theta}{\theta}$.

Table 1 shows values of θ and $\frac{\sin \theta}{\theta}$ as θ becomes smaller.

θ	$\sin \theta$	$\frac{\sin \theta}{\theta}$
1	0.84147	0.84147
0.1	0.09983	0.99833
0.01	0.00999	0.99983

Table 1: The value of $\frac{\sin \theta}{\theta}$ as θ tends to zero is 1.

You should verify these results with your calculator to appreciate that the value of $\frac{\sin \theta}{\theta}$ approaches 1 as θ tends to zero.

We now use these results in order to differentiate $f(x) = \sin x$ from first principles.

2. Differentiating $f(x) = \sin x$

Here $f(x) = \sin x$ so that $f(x + \delta x) = \sin(x + \delta x)$.

So

$$f(x + \delta x) - f(x) = \sin(x + \delta x) - \sin x$$

The right hand side is the difference of two sine terms. We use the first trigonometric identity (above) to write this in an alternative form.

$$\begin{aligned}\sin(x + \delta x) - \sin x &= 2 \cos \frac{x + \delta x + x}{2} \sin \frac{\delta x}{2} \\ &= 2 \cos \frac{2x + \delta x}{2} \sin \frac{\delta x}{2} \\ &= 2 \cos\left(x + \frac{\delta x}{2}\right) \sin \frac{\delta x}{2}\end{aligned}$$

Then, using the definition of the derivative

$$\begin{aligned}\frac{dy}{dx} &= \lim_{\delta x \rightarrow 0} \frac{f(x + \delta x) - f(x)}{\delta x} \\ &= \frac{2 \cos\left(x + \frac{\delta x}{2}\right) \sin \frac{\delta x}{2}}{\delta x}\end{aligned}$$

The factor of 2 can be moved into the denominator as follows, in order to write this in an alternative form:

$$\begin{aligned}\frac{dy}{dx} &= \frac{\cos\left(x + \frac{\delta x}{2}\right) \sin \frac{\delta x}{2}}{\delta x/2} \\ &= \cos\left(x + \frac{\delta x}{2}\right) \frac{\sin \frac{\delta x}{2}}{\frac{\delta x}{2}}\end{aligned}$$

We now let δx tend to zero. Consider the term $\frac{\sin \frac{\delta x}{2}}{\frac{\delta x}{2}}$ and use the result that $\lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} = 1$ with

$\theta = \frac{\delta x}{2}$. We see that

$$\lim_{\delta x \rightarrow 0} \frac{\sin \frac{\delta x}{2}}{\frac{\delta x}{2}} = 1$$

Further,

$$\lim_{\delta x \rightarrow 0} \cos\left(x + \frac{\delta x}{2}\right) = \cos x$$

So finally,

$$\frac{dy}{dx} = \cos x$$

3. The derivative of $f(x) = \cos x$.

Here $f(x) = \cos x$ so that $f(x + \delta x) = \cos(x + \delta x)$.

So

$$f(x + \delta x) - f(x) = \cos(x + \delta x) - \cos x$$

The right hand side is the difference of two cosine terms. This time we use the trigonometric identity

$$\cos C - \cos D = -2 \sin\left(\frac{C + D}{2}\right) \sin\left(\frac{C - D}{2}\right)$$

to write this in an alternative form.

$$\begin{aligned} \cos(x + \delta x) - \cos x &= -2 \sin \frac{x + \delta x + x}{2} \sin \frac{\delta x}{2} \\ -2 \sin \frac{2x + \delta x}{2} \sin \frac{\delta x}{2} \\ &= -2 \sin\left(x + \frac{\delta x}{2}\right) \sin \frac{\delta x}{2} \end{aligned}$$

Then, using the definition of the derivative

$$\begin{aligned} \frac{dy}{dx} &= \lim_{\delta x \rightarrow 0} \frac{f(x + \delta x) - f(x)}{\delta x} \\ &= \frac{-2 \sin\left(x + \frac{\delta x}{2}\right) \sin \frac{\delta x}{2}}{\delta x} \end{aligned}$$

The factor of 2 can be moved as before, in order to write this in an alternative form:

$$\begin{aligned} \frac{dy}{dx} &= -\frac{\sin\left(x + \frac{\delta x}{2}\right) \sin \frac{\delta x}{2}}{\delta x/2} \\ &= -\sin\left(x + \frac{\delta x}{2}\right) \frac{\sin \frac{\delta x}{2}}{\frac{\delta x}{2}} \end{aligned}$$

We now want to let δx tend to zero. As before

$$\lim_{\delta x \rightarrow 0} \frac{\sin \frac{\delta x}{2}}{\frac{\delta x}{2}} = 1$$

Further,

$$\lim_{\delta x \rightarrow 0} -\sin\left(x + \frac{\delta x}{2}\right) = -\sin x$$

So finally,

$$\frac{dy}{dx} = -\sin x$$

So, we have used differentiation from first principles to find the derivatives of the functions $\sin x$ and $\cos x$.