Engineering Maths First Aid Kit

8.9

Evaluating definite integrals

Introduction

Definite integrals can be recognised by numbers written to the upper and lower right of the integral sign. This leaflet explains how to evaluate definite integrals.

1. Definite integrals

The quantity

$$\int_a^b f(x)\,\mathrm{d}x$$

is called the **definite integral** of $f(x)$ from a to b. The numbers a and b are known as the **lower** and **upper limits** of the integral. To see how to evaluate a definite integral consider the following example.

Example

Find $\displaystyle\int_1^4 x^2\mathrm{d}x$.

Solution

First of all the integration of x^2 is performed in the normal way. However, to show we are dealing with a definite integral, the result is usually enclosed in square brackets and the limits of integration are written on the right bracket:

$$\int_1^4 x^2\,\mathrm{d}x = \left[\frac{x^3}{3}+c\right]_1^4$$

Then, the quantity in the square brackets is evaluated, first by letting x take the value of the upper limit, then by letting x take the value of the lower limit. The difference between these two results gives the value of the definite integral:

$$\begin{aligned}
\left[\frac{x^3}{3}+c\right]_1^4 &= \text{(evaluate at upper limit)} - \text{(evaluate at lower limit)}\\
&= \left(\frac{4^3}{3}+c\right) - \left(\frac{1^3}{3}+c\right)\\
&= \frac{64}{3} - \frac{1}{3}\\
&= 21
\end{aligned}$$

Note that the constants of integration cancel out. This will always happen, and so in future we can ignore them when we are evaluating definite integrals.

Example

Find $\int_{-2}^{3} x^3 \mathrm{d}x$.

Solution

$$\begin{aligned}
\int_{-2}^{3} x^3 \mathrm{d}x &= \left[\frac{x^4}{4}\right]_{-2}^{3} \\
&= \left(\frac{(3)^4}{4}\right) \quad - \quad \left(\frac{(-2)^4}{4}\right) \\
&= \frac{81}{4} - \frac{16}{4} \\
&= \frac{65}{4} \\
&= 16.25
\end{aligned}$$

Example

Find $\int_{0}^{\pi/2} \cos x \, \mathrm{d}x$.

Solution

$$\begin{aligned}
\int_{0}^{\pi/2} \cos x \, \mathrm{d}x &= [\sin x]_0^{\pi/2} \\
&= \sin\left(\frac{\pi}{2}\right) - \sin 0 \\
&= 1 - 0 \\
&= 1
\end{aligned}$$

Exercises

1. Evaluate

a) $\int_0^1 x^2 \mathrm{d}x$, b) $\int_2^3 \frac{1}{x^2} \mathrm{d}x$, c) $\int_1^2 x^2 \mathrm{d}x$, d) $\int_0^4 x^3 \mathrm{d}x$, e) $\int_{-1}^1 x^3 \mathrm{d}x$.

2. Evaluate $\int_3^4 x + 7x^2 \, \mathrm{d}x$.

3. Evaluate a) $\int_0^1 \mathrm{e}^{2x} \mathrm{d}x$, b) $\int_0^2 \mathrm{e}^{-x} \mathrm{d}x$, c) $\int_{-1}^1 x^2 \mathrm{d}x$, d) $\int_{-1}^1 5x^3 \mathrm{d}x$.

4. Find $\int_0^{\pi/2} \sin x \mathrm{d}x$.

Answers

1. a) $\frac{1}{3}$, b) $\frac{1}{6}$, c) $\frac{7}{3}$, d) 64, e) 0.

2. 89.833 (3dp).

3. a) $\frac{\mathrm{e}^2}{2} - \frac{1}{2} = 3.195$, (3dp), b) $1 - \mathrm{e}^{-2} = 0.865$ (3dp), c) $\frac{2}{3}$, d) 0.

4. 1.

