Maxima and minima

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In this unit we show how differentiation can be used to find the maximum and minimum values of a function.

Because the derivative provides information about the gradient or slope of the graph of a function we can use it to locate points on a graph where the gradient is zero. We shall see that such points are often associated with the largest or smallest values of the function, at least in their immediate locality. In many applications, a scientist, engineer, or economist for example, will be interested in such points for obvious reasons such as maximising power, or profit, or minimising losses or costs.

In order to master the techniques explained here it is vital that you undertake plenty of practice exercises so that they become second nature.

After reading this text, and/or viewing the video tutorial on this topic, you should be able to:

- use differentiation to locate points where the gradient of a graph is zero
- locate stationary points of a function
- distinguish between maximum and minimum turning points using the second derivative test
- distinguish between maximum and minimum turning points using the first derivative test

Contents

1. Introduction

In this unit we show how differentiation can be used to find the maximum and minimum values of a function. Because the derivative provides information about the gradient or slope of the graph of a function we can use it to locate points on a graph where the gradient is zero. We shall see that such points are often associated with the largest or smallest values of the function, at least in their immediate locality. In many applications, a scientist, engineer, or economist for example, will be interested in such points for obvious reasons such as maximising power, or profit, or minimising losses or costs.

2. Stationary points

When using mathematics to model the physical world in which we live, we frequently express physical quantities in terms of **variables**. Then, **functions** are used to describe the ways in which these variables change. A scientist or engineer will be interested in the ups and downs of a function, its maximum and minimum values, its turning points. Drawing a graph of a function using a graphical calculator or computer graph plotting package will reveal this behaviour, but if we want to know the precise location of such points we need to turn to algebra and differential calculus. In this section we look at how we can find maximum and minimum points in this way.

Consider the graph of the function, $y(x)$, shown in Figure 1. If, at the points marked A, B and C, we draw tangents to the graph, note that these are parallel to the x axis. They are horizontal. This means that at each of the points A, B and C the gradient of the graph is zero.

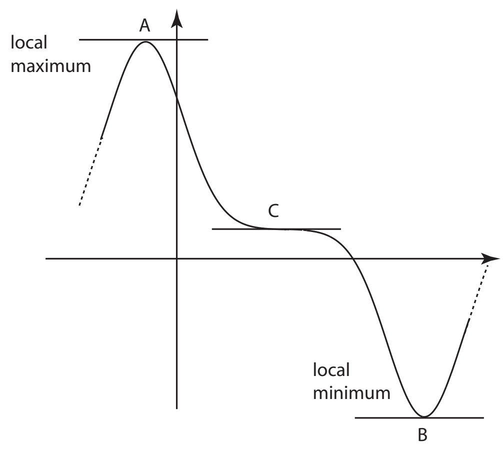

Figure 1. The gradient of this graph is zero at each of the points A, B and C.

We know that the gradient of a graph is given by $\frac{\mathrm{d}y}{\mathrm{d}x}$ Consequently, $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$ at points A, B and C. All of these points are known as **stationary points**.

Key Point

Any point at which the tangent to the graph is horizontal is called a **stationary point**.

We can locate stationary points by looking for points at which $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$.

3. Turning points

Refer again to Figure 1. Notice that at points A and B the curve actually turns. These two stationary points are referred to as **turning points**. Point C is not a turning point because, although the graph is flat for a short time, the curve continues to go down as we look from left to right.

So, all turning points are stationary points.

But not all stationary points are turning points (e.g. point C).

In other words, there are points for which $\frac{dy}{dx} = 0$ which are not turning points.

Key Point

At a **turning point** $\frac{dy}{dx} = 0$.

Not all points where $\frac{dy}{dx} = 0$ are turning points, i.e. not all stationary points are turning points.

Point A in Figure 1 is called a **local maximum** because in its immediate area it is the highest point, and so represents the greatest or maximum value of the function. Point B in Figure 1 is called a **local minimum** because in its immediate area it is the lowest point, and so represents the least, or minimum, value of the function. Loosely speaking, we refer to a local maximum as simply a **maximum**. Similarly, a local minimum is often just called a **minimum**.

4. Distinguishing maximum points from minimum points

Think about what happens to the gradient of the graph as we travel through the minimum turning point, from left to right, that is as x increases. Study Figure 2 to help you do this.

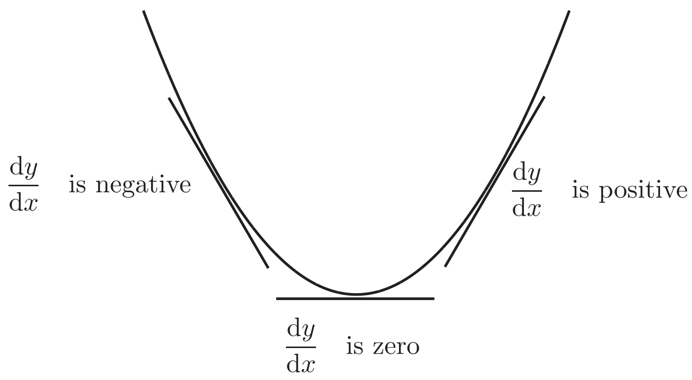

Figure 2. $\frac{dy}{dx}$ goes from negative through zero to positive as x increases.

Notice that to the left of the minimum point, $\frac{\mathrm{d}y}{\mathrm{d}x}$ is negative because the tangent has negative gradient. At the minimum point, $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$. To the right of the minimum point $\frac{\mathrm{d}y}{\mathrm{d}x}$ is positive, because here the tangent has a positive gradient. So, $\frac{\mathrm{d}y}{\mathrm{d}x}$ goes from negative, to zero, to positive as x increases. In other words, $\frac{\mathrm{d}y}{\mathrm{d}x}$ must be increasing as x increases.

In fact, we can use this observation, once we have found a stationary point, to check if the point is a minimum. If $\frac{\mathrm{d}y}{\mathrm{d}x}$ is increasing near the stationary point then that point must be minimum. Now, if the derivative of $\frac{\mathrm{d}y}{\mathrm{d}x}$ is positive then we will know that $\frac{\mathrm{d}y}{\mathrm{d}x}$ is increasing; so we will know that the stationary point is a minimum. Now the derivative of $\frac{\mathrm{d}y}{\mathrm{d}x}$, called the **second derivative**, is written $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$. We conclude that if $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$ is positive at a stationary point, then that point must be a minimum turning point.

Key Point

if $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$ at a point, and if $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} > 0$ there, then that point must be a minimum.

It is important to realise that this test for a minimum is not conclusive. It is possible for a stationary point to be a minimum even if $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$ *equals* 0, although we cannot be certain: other types of behaviour are possible. (However, we cannot have a minimum if $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$ is negative.)

To see this consider the example of the function $y = x^4$. A graph of this function is shown in Figure 3. There is clearly a minimum point when $x = 0$. But $\frac{\mathrm{d}y}{\mathrm{d}x} = 4x^3$ and this is clearly zero when $x = 0$. Differentiating again $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 12x^2$ which is also zero when $x = 0$.

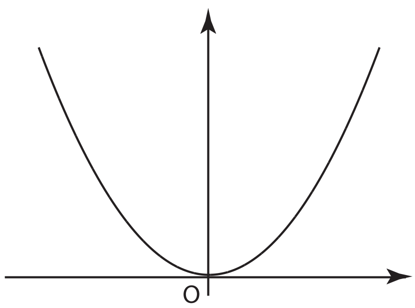

Figure 3. The function $y = x^4$ has a minimum at the origin where $x = 0$, but $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 0$ and so is not greater than 0.

Now think about what happens to the gradient of the graph as we travel through the maximum turning point, from left to right, that is as x increases. Study Figure 4 to help you do this.

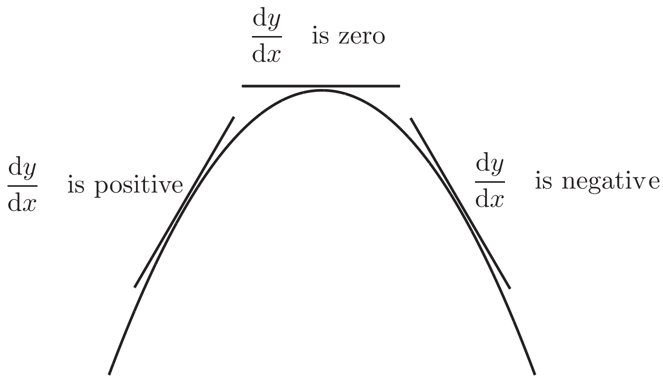

Figure 4. $\frac{\mathrm{d}y}{\mathrm{d}x}$ goes from positive through zero to negative as x increases.

Notice that to the left of the maximum point, $\frac{\mathrm{d}y}{\mathrm{d}x}$ is positive because the tangent has positive gradient. At the maximum point, $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$. To the right of the maximum point $\frac{\mathrm{d}y}{\mathrm{d}x}$ is negative, because here the tangent has a negative gradient. So, $\frac{\mathrm{d}y}{\mathrm{d}x}$ goes from positive, to zero, to negative as x increases.

In fact, we can use this observation to check if a stationary point is a maximum. If $\frac{\mathrm{d}y}{\mathrm{d}x}$ is decreasing near a stationary point then that point must be maximum. Now, if the derivative of $\frac{\mathrm{d}y}{\mathrm{d}x}$ is negative then we will know that $\frac{\mathrm{d}y}{\mathrm{d}x}$ is decreasing; so we will know that the stationary point is a maximum. As before, the derivative of $\frac{\mathrm{d}y}{\mathrm{d}x}$, the **second derivative** is $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$. We conclude that if $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$ is negative at a stationary point, then that point must be a maximum turning point.

Key Point

if $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$ at a point, and if $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} < 0$ there, then that point must be a maximum.

It is important to realise that this test for a maximum is not conclusive. It is possible for a stationary point to be a maximum even if $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 0$, although we cannot be certain: other types of behaviour are possible. But we cannot have a maximum if $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} > 0$, because, as we have already seen the point would be a minimum.

Key Point

The second derivative test: summary

We can locate the position of stationary points by looking for points where $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$.

As we have seen, it is possible that some such points will not be turning points.

We can calculate $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$ at each point we find.

If $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$ is positive then the stationary point is a minimum turning point.

If $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$ is negative, then the point is a maximum turning point.

If $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 0$ it is possible that we have a maximum, or a minimum, or indeed other sorts of behaviour. So if $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 0$ this second derivative test does not give us useful information and we must seek an alternative method (see Section 5).

Example

Suppose we wish to find the turning points of the function $y = x^3 - 3x + 2$ and distinguish between them.

We need to find where the turning points are, and whether we have maximum or minimum points. First of all we carry out the differentiation and set $\frac{\mathrm{d}y}{\mathrm{d}x}$ equal to zero. This will enable us to look for any stationary points, including any turning points.

$$\begin{aligned} y &= x^3 - 3x + 2 \\ \frac{\mathrm{d}y}{\mathrm{d}x} &= 3x^2 - 3 \end{aligned}$$

At stationary points, $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$ and so

$$\begin{aligned} 3x^2 - 3 &= 0 \\ 3(x^2 - 1) &= 0 \qquad \text{(factorising)} \\ 3(x-1)(x+1) &= 0 \qquad \text{(factorising the difference of two squares)} \end{aligned}$$

It follows that either $x - 1 = 0$ or $x + 1 = 0$ and so either $x = 1$ or $x = -1$.

We have found the x coordinates of the points on the graph where $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$, that is the stationary points. We need the y coordinates which are found by substituting the x values in the original function $y = x^3 - 3x + 2$.

<u>when $x = 1$</u>: $\quad y = 1^3 - 3(1) + 2 = 0$.

<u>when $x = -1$</u>: $\quad y = (-1)^3 - 3(-1) + 2 = 4$.

To summarise, we have located two stationary points and these occur at $(1, 0)$ and $(-1, 4)$.

Next we need to determine whether we have maximum or minimum points, or possibly points such as C in Figure 1 which are neither maxima nor minima.

We have seen that the first derivative $\frac{dy}{dx} = 3x^2 - 3$. Differentiating this we can find the second derivative:

$$\frac{d^2y}{dx^2} = 6x$$

We now take each point in turn and use our test.

<u>when $x = 1$</u>: $\frac{d^2y}{dx^2} = 6x = 6(1) = 6$. We are not really interested in this value. What is important is its sign. Because it is positive we know we are dealing with a minimum point.

<u>when $x = -1$</u>: $\frac{d^2y}{dx^2} = 6x = 6(-1) = -6$. Again, what is important is its sign. Because it is negative we have a maximum point.

Finally, to finish this off we produce a quick sketch of the function now that we know the precise locations of its two turning points (See Figure 5).

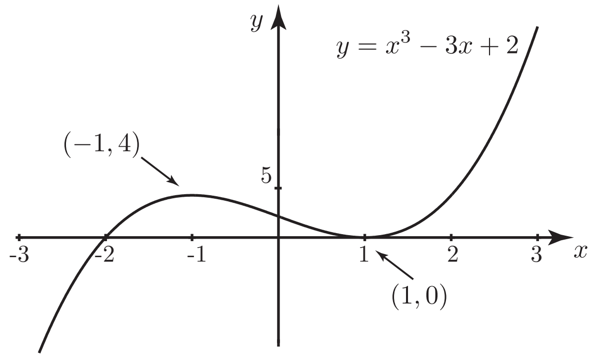

Figure 5. Graph of $y = x^3 - 3x + 2$ showing the turning points

5. An example which uses the first derivative to distinguish maxima and minima

Example

Suppose we wish to find the turning points of the function $y = \frac{(x-1)^2}{x}$ and distinguish between them.

First of all we need to find $\frac{dy}{dx}$.

In this case we need to apply the quotient rule for differentiation.

$$\frac{dy}{dx} = \frac{x \cdot 2(x-1) - (x-1)^2 \cdot 1}{x^2}$$

This does look complicated. Don't rush to multiply it all out if you can avoid it. Instead, look for common factors, and tidy up the expression.

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{x\cdot 2(x-1)-(x-1)^2\cdot 1}{x^2}\\ &= \frac{(x-1)(2x-(x-1))}{x^2}\\ &= \frac{(x-1)(x+1)}{x^2}\end{aligned}$$

We now set $\frac{\mathrm{d}y}{\mathrm{d}x}$ equal to zero in order to locate the stationary points including any turning points.

$$\frac{(x-1)(x+1)}{x^2} = 0$$

When equating a fraction to zero, it is the top line, the numerator, which must equal zero. Therefore

$$(x-1)(x+1) = 0$$

from which $x-1=0$ or $x+1=0$, and from these equations we find that $x=1$ or $x=-1$.

The y co-ordinates of the stationary points are found from $y = \frac{(x-1)^2}{x}$.

<u>when $x=1$</u>: $\quad y=0$.

<u>when $x=-1$</u>: $\quad y = \frac{(-2)^2}{-1} = -4$.

We conclude that stationary points occur at $(1,0)$ and $(-1,-4)$.

We now have to decide whether these are maximum points or minimum points. We could calculate $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$ and use the second derivative test as in the previous example. This would involve differentiating $\frac{(x-1)(x+1)}{x^2}$ which is possible but perhaps rather fearsome! Is there an alternative way ? The answer is yes. We can look at how $\frac{\mathrm{d}y}{\mathrm{d}x}$ changes as we move through the stationary point. In essence, we can find out what happens to $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$ without actually calculating it.

First consider the point at $x=-1$. We look at what is happening a little bit before the point where $x=-1$, and a little bit afterwards. Often we express the idea of 'a little bit before' and 'a little bit afterwards' in the following way. We can write $-1-\epsilon$ to represent a little bit less than -1, and $-1+\epsilon$ to represent a little bit more. The symbol ϵ is the Greek letter epsilon. It represents a small positive quantity, say 0.1. Then $-1-\epsilon$ would be -1.1, just a little less than -1. Similarly $-1+\epsilon$ would be -0.9, just a little more than -1.

We now have a look at $\frac{\mathrm{d}y}{\mathrm{d}x}$; not its value, but its sign.

When $x=-1-\epsilon$, say -1.1, $\frac{\mathrm{d}y}{\mathrm{d}x}$ is positive.

When $x=-1$ we already know that $\frac{\mathrm{d}y}{\mathrm{d}x}$=0.

When $x = -1 + \epsilon$, say -0.9, $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ is negative.

We can summarise this information as shown in Figure 6.

	$x = -1 - \epsilon$	$x = -1$	$x = -1 + \epsilon$
sign of $\dfrac{\mathrm{d}y}{\mathrm{d}x}$	$+$	0	$-$
shape of graph	$\nearrow$	$\longrightarrow$	$\searrow$

Figure 6. Behaviour of the graph near the point $(-1, -4)$

Figure 6 shows us that the stationary point at $(-1, -4)$ is a maximum turning point. Then we turn to the point $(1, 0)$. We carry out a similar analysis, looking at the sign of $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ at $x = 1 - \epsilon$, $x = 1$, and $x = 1 + \epsilon$. The results are summarised in Figure 7.

	$x = 1 - \epsilon$	$x = 1$	$x = 1 + \epsilon$
sign of $\dfrac{\mathrm{d}y}{\mathrm{d}x}$	$-$	0	$+$
shape of graph	$\searrow$	$\longrightarrow$	$\nearrow$

Figure 7. Behaviour of the graph near the point $(1, 0)$

We see that the point is a minimum.

This, so-called **first derivative test**, is also the way to do it if $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}$ is zero in which case the second derivative test does not work. Finally, for completeness a graph of $y = \dfrac{(x-1)^2}{x}$ is shown in Figure 8 where you can see the maximum and minimum points.

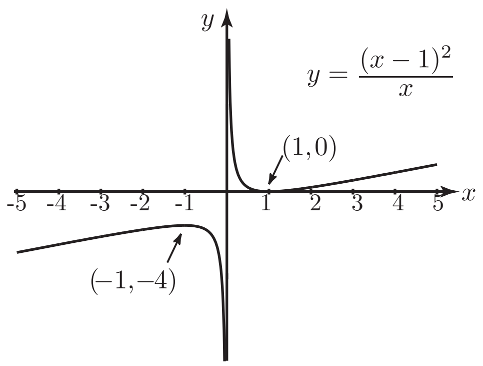

Figure 8. A graph of $y = \dfrac{(x-1)^2}{x}$ showing the turning points

Exercises

Locate the position and nature of any turning points of the following functions.

1. $y = \frac{1}{2}x^2 - 2x$, 2. $y = x^2 + 4x + 1$, 3. $y = 12x - 2x^2$, 4. $y = -3x^2 + 3x + 1$,
5. $y = x^4 + 2$, 6. $y = 7 - 2x^4$, 7. $y = 2x^3 - 9x^2 + 12x$, 8. $y = 4x^3 - 6x^2 - 72x + 1$,
9. $y = -4x^3 + 30x^2 - 48x - 1$, 10. $y = \dfrac{(x+1)^2}{x-1}$.

Answers

1. Minimum at $(2, -2)$, 2. Minimum at $(-2, -3)$, 3. Maximum at $(-3, -54)$,

4. Maximum at $(\frac{1}{2}, \frac{7}{4})$, 5. Minimum at $(0, 2)$, 6. Maximum at $(0, 7)$,

7. Maximum at $(1, 5)$, minimum at $(2, 4)$, 8. Maximum at $(-2, 89)$, minimum at $(3, -161)$,

9. Maximum at $(4, 31)$, minimum at $(1, -23)$, 10. Maximum at $(-1, 0)$, minimum at $(3, 8)$.