Integration: Laplace Transforms

mccp-Fletcher-006-v2

Suppose $f(t)$ is a function of time defined for $t \geq 0$. Its *Laplace Transform*, which has a wide range of applications in engineering, is the function $F(s)$ of a new variable s defined by

$$\mathcal{L}(f) = F(s) = \int_0^\infty f(t)e^{-st}\,\mathrm{d}t$$

This note reviews the techniques of integration needed to find and manipulate Laplace Transforms.

Powers

Example If $f(t) = t$ then

$$\begin{aligned}
F(s) = \mathcal{L}(t) &= \int_0^\infty te^{-st}\,\mathrm{d}t \\
&= \underbrace{\left[t \times \frac{-1}{s}e^{-st}\right]_{t=0}^{t=\infty}}_{=0 \text{ for } t=0 \text{ and } t=\infty} - \int_0^\infty 1 \times \frac{-1}{s}e^{-st}\,\mathrm{d}t \quad \text{integrating by parts} \\
&= -\left[\frac{1}{s^2}e^{-st}\right]_{t=0}^{t=\infty} \quad \text{integrating again, noting three minus signs} \\
&= \frac{1}{s^2} \quad \text{substituting limits } t=\infty \text{ and } t=0
\end{aligned}$$

Exercise Use integration by parts to show that $\mathcal{L}\left(t^2\right) = (2/s) \times \mathcal{L}(t)$. Generalise this to $\mathcal{L}\left(t^n\right)$.

Exponential and trigonometric functions

Example

$$\mathcal{L}\left(e^{-at}\right) = \int_0^\infty e^{-at}e^{-st}\,\mathrm{d}t = \int_0^\infty e^{-(s+a)t}\,\mathrm{d}t = \left[-\frac{1}{s+a}e^{-(s+a)t}\right]_{t=0}^{t=\infty} = \frac{1}{s+a}$$

This example with $a = -j\omega$ can be used to find $\mathcal{L}(\cos\omega t)$ and $\mathcal{L}(\sin\omega t)$:

$$\begin{aligned}
\mathcal{L}\left(e^{j\omega t}\right) &= \frac{1}{s-j\omega} = \frac{s+j\omega}{(s+j\omega)(s-j\omega)} \quad \text{to make the denominator real} \\
&= \frac{s+j\omega}{s^2+\omega^2} = \frac{s}{s^2+\omega^2} + j\frac{\omega}{s^2+\omega^2}
\end{aligned}$$

Exercise Compare the imaginary parts of Euler's formula $\cos(\omega t) + j\sin(\omega t) = e^{j\omega t}$ and the final expression here to show that $\mathcal{L}(\sin(\omega t)) = \omega/(s^2+\omega^2)$. What is $\mathcal{L}(\cos(\omega t))$?

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Other types of function

Example The Laplace transform of the function $f(t)$ defined by

$$f(t) = \begin{cases} t & \text{if } 0 \le t \le 1 \\ 1 & \text{if } t \ge 1 \end{cases}$$

Figure 1: Graph of $f(t)$

is $(1-e^{-s})/s^2$ as this calculation shows:

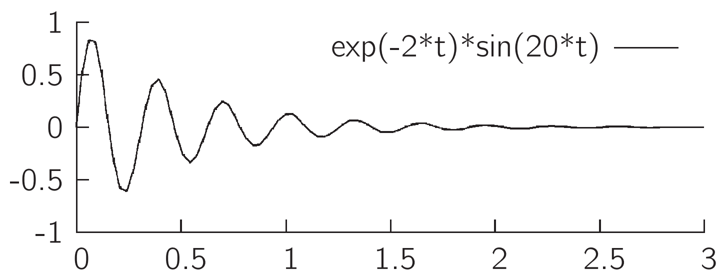
$$\begin{aligned}
\mathcal{L}(f(t)) &= \int_0^1 t \times e^{-st}\,\mathrm{d}t \quad \text{using the definition of } f(t) \text{ for } 0 \le t \le 1 \\
&\quad + \int_1^\infty 1 \times e^{-st}\,\mathrm{d}t \quad \text{using the definition of } f(t) \text{ for } t \ge 1 \\
&= \underbrace{\left[t \times \frac{-1}{s}e^{-st}\right]_{t=0}^{t=1}}_{\text{(a)}} - \underbrace{\int_0^1 1 \times \frac{-1}{s}e^{-st}\,\mathrm{d}t}_{\text{(b)}} \quad \text{integrating by parts for the first integral} \\
&\quad + \underbrace{\left[\frac{-1}{s}e^{-st}\right]_{t=1}^{t=\infty}}_{\text{(c)}} \quad \text{evaluating the second integral} \\
&= \frac{-e^{-s}}{s} \quad \text{substituting limits } t=1 \text{ and } t=0 \text{ in (a)} \\
&\quad \underbrace{-\left[\frac{1}{s^2}e^{-st}\right]_{t=0}^{t=1}}_{\text{(d)}} \quad \text{integrating (b) again, noting three minus signs} \\
&\quad + \frac{e^{-s}}{s} \quad \text{substituting limits } t=\infty \text{ and } t=1 \text{ in (c)} \\
&= \frac{-e^{-s}}{s} + \frac{e^{-s}}{s} \quad \text{already found and cancelling out} \\
&\quad + \frac{1-e^{-s}}{s^2} \quad \text{substituting limits } t=1 \text{ and } t=0 \text{ in (d), giving } \mathcal{L}(f(t))
\end{aligned}$$

The First Shift Theorem

This theorem says that if $\mathcal{L}(f(t)) = F(s)$ then $\mathcal{L}(f(t)e^{-at}) = F(s+a)$. To see this compare these integrals; the second is similar to the first, but with s replaced by $s+a$.

$$\begin{aligned}
\mathcal{L}(f(t)) &= \int_0^\infty f(t)e^{-st}\,\mathrm{d}t \\
\mathcal{L}(f(t)e^{-at}) &= \int_0^\infty f(t)\underbrace{e^{-at}e^{-st}}_{=e^{-(s+a)t}}\,\mathrm{d}t
\end{aligned}$$

Figure 2: Decaying sinusoidal oscilliations



Example The Laplace transform of the decaying sinusoidal oscillations $e^{-2t}\sin(20t)$ is $\dfrac{20}{(s+2)^2+400}$

Exercise What is the Laplace Transform of $e^{-at}\cos(\omega t)$? **Answer** $\dfrac{s+a}{(s+a)^2+\omega^2}$

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